A MODEL-BASED ACOUSTIC SOURCE LOCALIZATION USING THE MDOF TRANSMISSIBILITY CONCEPT

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Abstract
Here is proposed a generalization of the acoustic pressure transmissibility to multiple degrees-of-freedom (MDOF) systems. The main objective is to obtain simultaneously the pressure at several coordinates (unknown pressures) through a set of measured pressures (known pressures) from other points. It can be advantageous, e.g., if the unknown pressures are in locations of difficult access. The proposed concept of MDOF acoustic transmissibility is used to develop a simple method to identify the location of acoustic harmonic sources in steady-state conditions by relating measured pressures against estimated pressures from a numerical or analytical model. In this work a finite element model is used. Simple examples are presented to illustrate the identification of punctual sources in one dimensional domains, using scalar and/or transmissibility matrix. The obtained results illustrate the potential and limitations of the proposed model-based acoustic source localization method.

Keywords: transmissibility matrix, linear acoustic, estimated pressure, source localization, finite element.

1   Introduction

This work is dedicated to a generalization of the acoustic pressure transmissibility to multiple-degrees-of-freedom (MDOF) systems and to its computational implementation for simple cases of acoustic source localization. This is a preliminary evaluation on the viability and potentialities (as well as limitations) of these two new developments.
Since previous developments from one of the authors [1] – namely in the use of the concept of force transmissibility and displacement transmissibility for the dynamical force identification – that its generalization to acoustic source localization was considered to be a natural although not a straightforward development. Pioneer work on operational acoustic modal analysis (OAMA) using transmissibility measurements was published by C. Devrient et al. [2]. In their work, they discuss the problem that available techniques presented several difficulties to correctly identify the acoustic parameters. For example, the existing experimental Acoustic Modal Analysis techniques use volume

acceleration sources while the OAMA technique not. But, an important disadvantage of existing OAMA, mentioned in [2], is that the non-measured acoustic sources must be pure white noise excitation which in operation cannot always be the case. In order to solve this requirement, they combined transmissibility measurements under different loading conditions and shown in their paper that acoustic parameters can be identified by that way for the presented example of an acoustic cavity.

In the present work, the main objective is to obtain simultaneously the pressure at several coordinates (unknown pressures) through a set of measured pressures (known pressures) from other points. It can be advantageous, e.g., if the unknown pressures are in locations of difficult access. The proposed concept of MDOF acoustic pressure transmissibility is used to develop a "simple" method to identify the location of acoustic harmonic sources by relating measured pressures against estimated pressures from a numerical or analytical model. In this work a finite element (FE) model is used. Simple examples are presented to illustrate the identification of punctual sources in one dimensional domains, using scalar and/or matrix transmissibility. The obtained results illustrate the potential and limitations of the proposed model-based acoustic source localization method. The techniques adopted have been successfully generalized by the authors to 2D acoustic problems.

Finally, note that this identification problem suffers – in common with other inverse problems – from the effects of matrix ill-conditioning as well as the ill-posedness of the inverse problem. Another consequence is the non-uniqueness of the solution.

2 Fundamentals

Like any linear dynamic undamped structural (upper index 's') mechanical systems can be modelled in the frequency-domain by the following steady-state equations:

$$\left(K^{\,s} - \omega^2 \cdot M^{\,s}\right) X(\omega) = F(\omega), \qquad (1)$$

where $K^{\,s}$ is the structural stiffness matrix, $M^{\,s}$ is the structural mass matrix, $X(\omega)$ the amplitude displacement vector, and $F(\omega)$ the amplitude force vector; the same can be set for any linear dynamic undamped fluid (upper index 'f') acoustic systems that can be modelled in the frequency-domain by:

$$\left(K^{\,f} - \omega^2 \cdot M^{\,f}\right) P(\omega) = \dot{Q}(\omega), \qquad (2)$$

where $K^{\,f}$ is the global acoustic stiffness matrix, $M^{\,f}$ is the global acoustic mass matrix, $P(\omega)$ the amplitude pressure vector, and $Q(\omega)$ the volume acceleration vector.

As mentioned in Devriendt et al. [2], one can conclude that similar frequency response functions (FRF) formula can be applied using the respective modal analysis. The same authors also mention that FRF are widely used in the field of experimental modal analysis (EMA) only in recent times the transmissibility functions made their arrival in the field of operational modal analysis (OMA) [2,3]. The transmissibility functions are the ratio between two signals like, e.g., the displacement transmissibility function that is the ratio of the motion response (output) by the motion excitation (input) or the force transmissibility function that is the ratio of the force response (output/reactions) by the force excitation (input). For a description on these concepts from structural vibrations see e.g. Lage et al. [1] and Maia et al. [4], and the references therein.

Ideas of pressure transmissibility are not new, and efforts in their development can be found, e.g., in [5] where the authors proposed a discrete method for the acoustic transmissibility of a pressure transducer-in-capsule used in experiments to measure wall pressure spectra with application to the vibration of nuclear reactor fuel rods and heat exchanger tubes.

In [2], it is introduced the pressure transmissibility function as the ratio of the pressure response (output) over the pressure (input) and refer that transmissibility functions are ratios of same type of signals as opposed to FRFs which are defined by a ratio between conjugate variables (motions response/force input or pressure response/volume acceleration input). One advantage of the

transmissibility functions is that they can be measured without the knowledge of the specific excitation forces or volume accelerations *q* of each situation.

Assuming an one-point acoustic source in the degree of freedom (DOF) *k*, the transmissibility function $T_{ir}(\omega)$ is the ratio between the two pressures $p_i(\omega)$ and $p_r(\omega)$, respectively measured at the DOFs *i* and *r*. Note that a careful consideration should be taken to the choice of the reference DOF *k* [2].

$$T_{ir}(\omega) = \frac{p_i(\omega)}{p_r(\omega)} = \frac{H_{ik}(\omega)q_k(\omega)}{H_{rk}(\omega)q_k(\omega)}. \tag{3}$$

Assuming multiple point acoustic sources in the DOFs *k=1,2,...,n*, the transmissibility $T_{ir}(\omega)$ is defined as the ratio between the two pressures $p_i(\omega)$ and $p_r(\omega)$, respectively measured at the DOFs *i* and *r*, i.e.

$$T_{ir}(\omega) = \frac{p_i(\omega)}{p_r(\omega)} = \frac{\sum_{k=1}^{n} H_{ik}(\omega)q_k(\omega)}{\sum_{k=1}^{n} H_{rk}(\omega)q_k(\omega)}. \tag{4}$$

3   Methodologies

Authors propose here two different techniques to obtain the acoustic pressure transmissibility. The first one uses a Dynamic Stiffness matrix formulation, while the second uses a FRF formulation.

3.1   Transmissibility from Dynamic Stiffness Matrix

As mentioned, linear dynamic acoustic systems can be modelled in the frequency-domain by the steady-state system (2). It can be presented in the following form:

$$[Z(\omega)]\{P(\omega)\} = \{F(\omega)\}, \tag{5}$$

where *Z* is the dynamic stiffness matrix given by $K - \omega^2 M + j\omega C$. Here $j = \sqrt{-1}$. In inviscid systems *C* is zero, but this matrix can be introduced due to impedance boundary conditions. *P* is the vector of nodal pressures and *F* is the vector of acoustic loads.

To obtain the pressure transmissibility, one can consider the following three sets of coordinates named as: 1) the set *U* of coordinates where pressure amplitude are imposed; 2) the set *K* of coordinates of where pressure is known; and 3) the set *C* of the remaining coordinates. Fig. 1 illustrates this division of the acoustic domain (the choice by the letter *U* can be justified by the fact that these locations will be the unknown locations in the localization of the acoustic sources).

Grouping all the coordinates in the three different sets *K, U* and *C*, and *F*={0,0,0} one obtains:

$$\begin{bmatrix} \mathbf{Z}_{KK} & \mathbf{Z}_{KU} & \mathbf{Z}_{KC} \\ \mathbf{Z}_{UK} & \mathbf{Z}_{UU} & \mathbf{Z}_{UC} \\ \mathbf{Z}_{CK} & \mathbf{Z}_{CU} & \mathbf{Z}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{P}_K \\ \mathbf{P}_U \\ \mathbf{P}_C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{6}$$

Regrouping the sets *K* and *C* in a new set *R*, then (6) results in the following system of equations.

$$\begin{bmatrix} \mathbf{Z}_{RR} & \mathbf{Z}_{RU} \\ \mathbf{Z}_{UR} & \mathbf{Z}_{UU} \end{bmatrix} \begin{Bmatrix} \mathbf{P}_R \\ \bar{P}_U \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{7}$$

When the imposed pressures, $\bar{P}_U$, are known we have the direct problem which is

$$\mathbf{Z}_{RR}\mathbf{P}_R = -\mathbf{Z}_{RU}\bar{P}_U, \tag{8}$$

The equation (8) allows to obtain the pressure transmissibility between the sets U and R, i.e.

$$\mathbf{P}_R = -(\mathbf{Z}_{RR})^{-1}\mathbf{Z}_{RU}\bar{P}_U \quad \Rightarrow \quad \mathbf{T}_{RU}^Z = -(\mathbf{Z}_{RR})^{-1}\mathbf{Z}_{RU} \quad \Leftrightarrow \quad \mathbf{T}_{RU}^Z = -(\mathbf{Z}_{UR})^+\mathbf{Z}_{UU}. \tag{9}$$
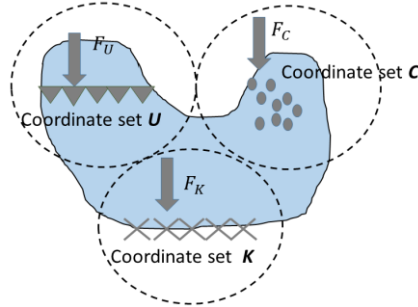

Figure 1 – Schematic illustration of the domain division in set U where pressures are imposed (sources); set K where pressures are known ("measured"); and set C with the remaining coordinates.

Note that in practice to extract the transmissibility between the sets U and K, only the k lines and columns of $\mathbf{T}_{RU}^Z$ are needed. For the pseudoinverse, the number of DOFs of U, i.e. $\#U$, has to be greater than the corresponding number of DOFs of R, i.e. $\#R$, which is not expected in practice.

3.2 Transmissibility from Frequency Response Matrix

An alternative to the methodology presented in section 3.1, is to use FRFs, i.e. $\boldsymbol{H}(\omega)$, which using the same division of the acoustic domain in the sets U, K and C can be presented in the following form:

$$\{P(\omega)\} = [H(\omega)]\{F(\omega)\} \Rightarrow \begin{Bmatrix} \mathbf{P}_K \\ \mathbf{P}_U \\ \mathbf{P}_C \end{Bmatrix} = \begin{bmatrix} \mathbf{H}_{KK} & \mathbf{H}_{KU} \\ \mathbf{H}_{UK} & \mathbf{H}_{UU} \\ \mathbf{H}_{CK} & \mathbf{H}_{CU} \end{bmatrix} \begin{Bmatrix} \mathbf{F}_K \\ \mathbf{F}_U \end{Bmatrix}, \tag{10}$$

where $\mathbf{F}_C = 0$ is assumed. From the first and second lines, one can obtain the following equations.

$$\mathbf{P}_K = \mathbf{H}_{KK}\mathbf{F}_K + \mathbf{H}_{KU}\mathbf{F}_U, \tag{11.a}$$

$$\bar{P}_U = \mathbf{H}_{UK}\mathbf{F}_K + \mathbf{H}_{UU}\mathbf{F}_U. \tag{11.b}$$

Solving for $\boldsymbol{F}_U$ in (11.b), substituting into (11.a), and assuming $\boldsymbol{F}_K = 0$ one gets the pressure transmissibility between the sets K and U, i.e.

$$\boldsymbol{T}_{KU}^H = \boldsymbol{H}_{KU}\left(\boldsymbol{H}_{UU}\right)^{-1}. \tag{12}$$

3.3 Methodology for the Localization of Punctual Sources in a One-dimensional Domain

The objective of this source localization is to estimate the positions of n acoustic sources, knowing the pressure at some measured points (or positions).

Figure 2 – One-dimensional acoustic domain, discretized in 15 coordinates, indicating the set U having one source pressure p_U (green triangle) and the set K having a measured pressure p_K (circle).

The basic idea behind the proposed source localization is to look for the possible set or sets of active sources that minimize, over a predefined range of frequencies, the "error" (difference) between the components of the measured pressures and the estimated pressures (upper index 's'). In the expression, *nf* is the number of frequencies in which the range of frequencies is discretized, and the estimated pressure is obtained through transmissibility matrix between K and U using a model of the system.

$$\grave{o}_P^s = \sum_{i=1}^{nf} \left[P_K(\omega_i) - P_K^s(\omega_i) \right]^2 . \Rightarrow \grave{o}_P^s = \sum_{i=1}^{nf} \left[P_K(\omega_i) - T_{KU}^s(\omega_i)\overline{P_U} \right]^2 \tag{13}$$

In this work, instead of the error (13), an average correlation coefficient between known and estimated pressure curves (which are functions of frequency) is used with n_K known pressures:

$$\overline{\sigma} = \frac{1}{n_K} \sum_{i=1}^{n_K} \frac{COV(P_{K_i}^s, P_{K_i})}{\sqrt{VAR(P_{K_i}^s).VAR(P_{K_i})}} . \tag{14}$$

This multiple correlation of P^s and P (estimated and known pressure) uses covariance (COV) and the variance (VAR). The use of this correlation assumes that the pressure amplitude |P| is used, the source is treated as imposed pressure amplitude; that in case of more than one source is used then all sources are in phase; the pressure curve for the given frequency range is known in a set of nodes K. The localization process works for any given pressure amplitude $\overline{P_U}$. The method requires a numbered list of combinations of nodes where the sources can be applied (in MATLAB, a nchoosek command can be used, returning all combinations of the given na nodes taken k sources at a time, successively k = 1, 2, 3,… The total number of combinations 'npt' is defined). The list of possible locations of the punctual source(s) starts with the "combinations" or positions number 1,2,3,4,5,….,*n* corresponding to the *n* coordinates (mesh nodes). It is followed by combinations of possible locations of two sources, which [1 2],[1 3],[1 4],…,[1 *n*]; [2 3],[2 4],[2 5],…,[2 *n*];…; [*n-1* n]. In this way the combination number n+1 is related with the positions [*1 2*], and the combination 2*n is given by the positions [*1 n*], and so on. The list finishes when the predefined maximum allowed number of sources is treated.

4   Results and Discussion

To illustrate the proposed methodologies, two verifications and two simple examples are presented.

4.1   FEA model verification

A rectilinear tube is modelled considering a plane wave and different boundary condition at the ends. Figure 3 illustrates a 3D representation and its simplified 2D axisymmetric model and 1D model.
Here, we reproduce briefly the FE study published in Cartaxo [7] on the numerical behavior of a sound wave propagation in a tube with dimensions Ø40mmx500mm and a harmonic pressure of amplitude P=1 Pa applied on left end (first coordinate/node). For the right end, two situations are studied: a) completely rigid top i.e. totally reflective top; and b) completely absorbing top.

The medium is at a reference pressure and the amplitude is the fluctuation around this mean value. While for the anechoic top the model does not present difficulty to achieve the correct wave response of the system, for the totally reflective top one can see (Fig. 4 a) that 45 elements were not enough to achieve the amplitude with the 1D FE [6] used here, while in [7] Cartaxo used 2D and 3D acoustic FE which required 36 elements per wavelength for the same problems.
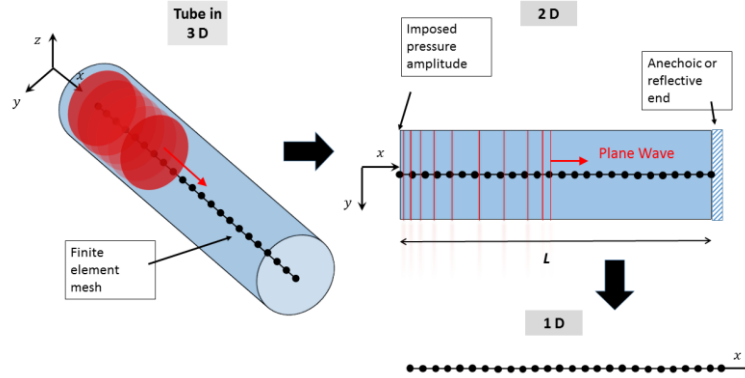

Figure 3 – Simplification of the 3D acoustic domain to a 1D domain discretized in 15 coordinates.

For a one-dimensional propagation, as within a straight duct, this equation can be simply written as:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0 \,. \tag{15}$$

It is a one-dimensional homogeneous partial differential equation with constant coefficients that has a general analytical solution given by [8]:

$$p(t,x) = [C_1 e^{-jkx} + C_2 e^{+jkx}]e^{j\omega t} \Rightarrow p(x) = [A\,e^{-jkx} + B\,e^{+jkx}]. \tag{16}$$

For a harmonic pressure, according to [8] one has, at x= l, A=p_0 and B=0 in case of an anechoic end. But in the case of a completely reflective end, A=1−B and B=(1/2)(1-j tan(kl)), and the analytical solution is:

$$p_{exact}(x) = p0\,[\cos(kx) + \tan(kl)\,\sin(kx)] \tag{17}$$

Both, steady-state solutions are plotted at Fig. 4. The pressure along the axis of length $L = 0.5$ m can be obtained from Eq. (17), considering p0= 1 Pa, and $k=(2\pi)/\lambda=\omega/c=2\pi f/c$ is obtained with f=1500 Hz and c=344 m/s. For the tube with a given radius r we need to verify that kr is less than 1.84.

Results show that to obtain a solution with an error less than ~ 1%, it required a 1D mesh with practically 80 or more elements per wave length. It is well-known – as described, e.g., in [6-8] – the difficulty in achieving an adequate accuracy with a standard FEM (_h_-FEM). Indeed, an insufficient number of elements per wavelength result in a badly modeled wave. It is also known as the Pollution Error that is given by the form $C_1 hk + C_2 k^3 h^2$, where the constants are independent of both h (characteristic length of the element) and k. It makes clear that an increased number of finite elements in a model do not guarantee the improvement in the solution accuracy.

Due to it, the simulation of wave propagation still is one of the most challenging issues in computational mechanics. Indeed, more accurate numerical solutions are needed and one key issue is the control of the pollution error [9] when higher frequencies are involved. A detailed discussion on this issue for the _h_-version or _h−p_ version of the Galerkin FEM can be found in [10,11]. So, in order to control that numerical error, it is suggested to reduce _h_ in such a way that $h_2 k^3$ remains as constant as possible, or to use higher order piecewise polynomials or basis functions that are specifically tailored to the high frequency. Such techniques, and discussion on it, are out of scope of this work.
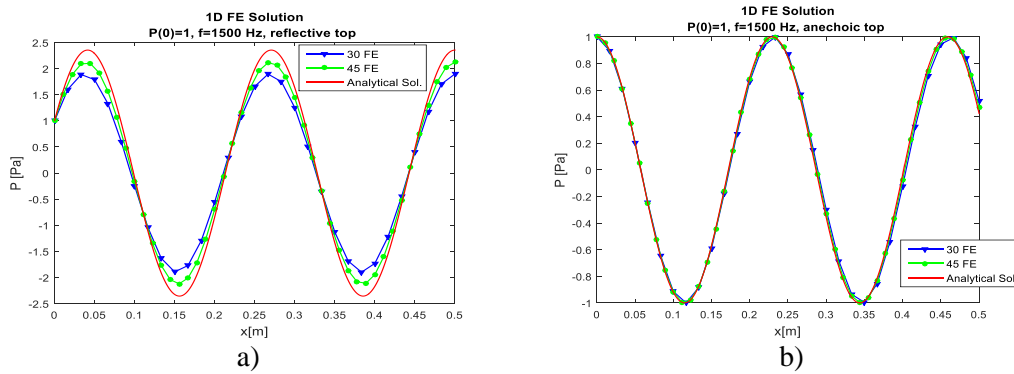
a)
b)

Figure 4 – Plots of the pressure in the 1D model of the tube, discretized in 31 and 46 coordinates (or nodes) versus the analytical solution for: a) a totally reflective top, and b) an anechoic top.

4.2 Verification of the scalar pressure transmissibility

The purpose here is to obtain the transmissibility functions for the same problem treated in [2]. The one dimensional domain is modelled (as in the section 4.1) with sixty 1D finite elements and a total length L = 1.24 m (Fig. 5). The acoustic source is located at x=0, and the reference pressure node is in the same location. Several transmissibility curves are plotted at Fig. 6 for different measure points *'i'*.
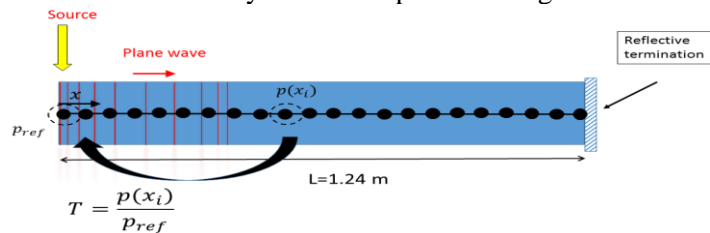
Figure 5 – Illustration of the tube with the acoustic source at one end and a reflective top at the other.

In Figure 6 a), one can observe curves that are similar to the ones obtained by [2], where the pressure transmissibility present peaks at different frequencies from the natural frequencies. These peaks occur at approximately the same frequencies as in [2]. The differences are due to the fact that, in this work a one dimensional mesh is used to model the acoustic medium (cavity), as opposite to [2] where a 3D mesh is used. Nevertheless, in both studies only the centerline of the domain is analyzed. Other influence factors are the mesh refinement and/or sound speed (*c*) value used in the calculations.
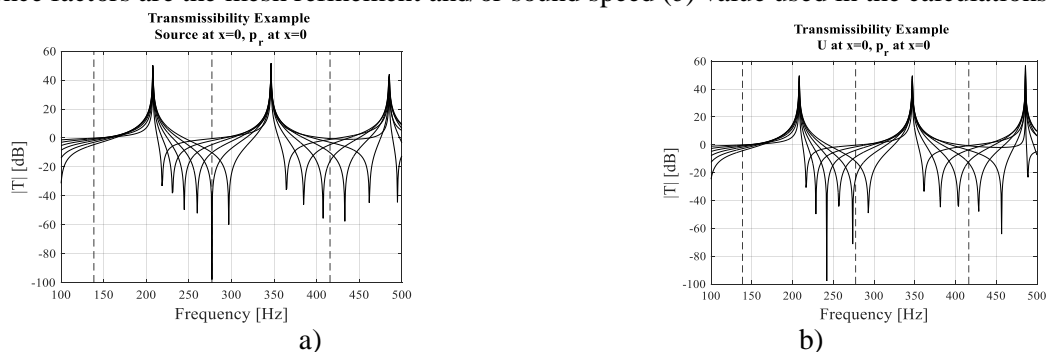

a)
b)

Figure 6 –Transmissibility plots: a) as obtained in [2]; and b) using proposed equation (12).

Figure 6 b) shows the same transmissibility curves but this time with an imposed pressure instead of a source (at x=0). It is clear that the curves in both plots are very similar, as expected.

4.3   Verification of the proposed pressure transmissibility matrix

From this point on, only the formulations proposed at the sections 3.2 and 3.3 are used. Also, for simplicity, from now on the tube length used is L=1 m. The concept introduced in those sections, opens the possibility to calculate a transmissibility matrix that relates a set K of pressures with another a set U of pressures. This concept can be used to locate multiple punctual sources, as e.g. illustrated in Fig. 7 where two U coordinates (extremes of the tube) with two K coordinates at x=L/4 and x=3L/4.
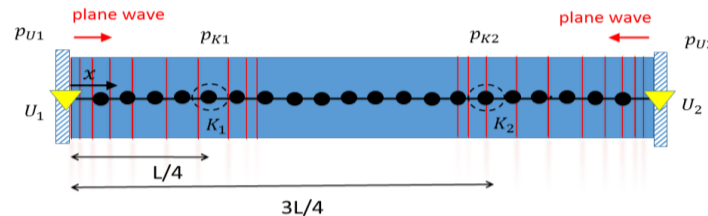

Figure 7 – Tube with two sources (triangle marks) and two measure points (circle marks).

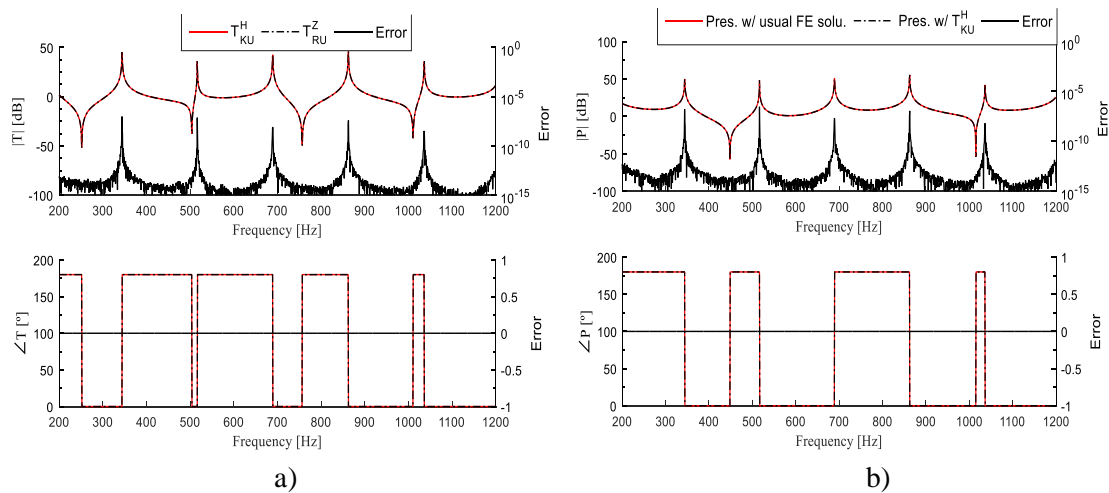a)   b)

Figure 8 – Plots of the amplitude and phase for: a) the transmissibility matrix between sets K and U ; and b) the pressure, at x=3L/4, obtained directly from FE solution vs obtained from $\boldsymbol{T}_{KU}^{H}$.

At Fig. 8 a) are presented the transmissibility curves obtained with $\boldsymbol{Z}$ as well as with $\boldsymbol{H}$, and the deviation for the T_{11} matrix entry, which is very low. Several verifications at distinct points were done, as e.g. between pressure values obtained via the transmissibility matrix and ones obtained from the standard FE solution (Fig.8 b). In all tested cases the error is associated with the numerical representation round-off errors and floating point operations.

4.4   Source Localization

4.4.1 Localization of one acoustic source

Here is presented an application of the proposed transmissibility matrix. As illustrated by the Fig. 9, a punctual harmonic source with *3 Pa* (amplitude) is to be identified using the data from a measurement performed at the right termination and the proposed FE-based localization methodology (section 3.3). The range of frequencies used to calculate the transmissibility is 200-1200 Hz. Fig. 10 presents the correlation plots having in horizontal axis the number of combination. In both, reflective and anechoic cases, the identification was successful i.e. the correct source location (in blue) coincides with the

combination that presents the maximum correlation (red square). In the correlation for anechoic cases, we still need to clarify why results (Fig. 10b) are better using the real part of P instead of the module.
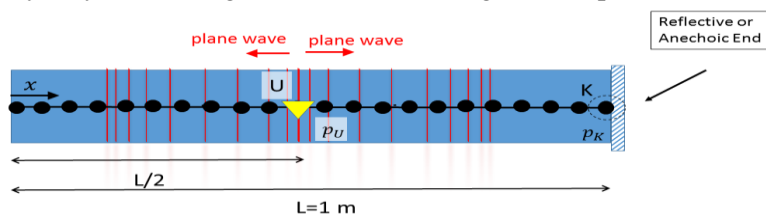

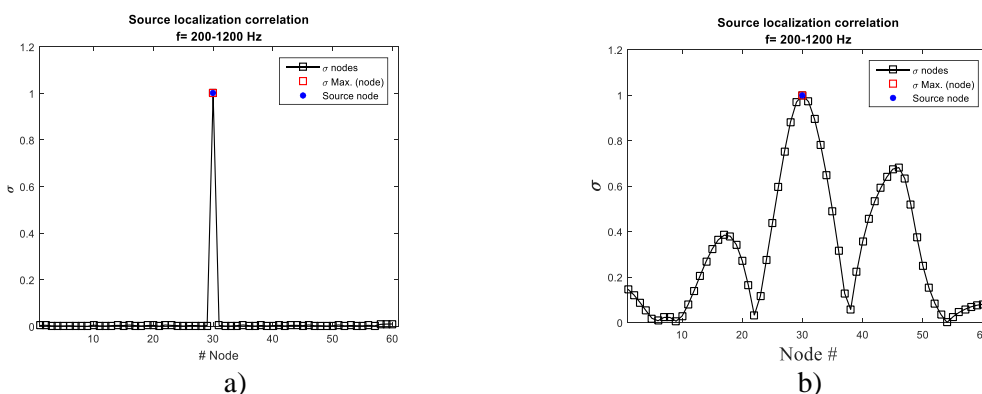Figure 9 – Schematics of 1D source identification

a)   b)

Figure 10 –Correlation vs combination number for: a) the reflective top; and b) the anechoic top.

4.4.2 Simultaneous localization of two acoustic sources

For the tube illustrated at Fig.7, consider a pressure (amplitude) of *3 Pa* and *1 Pa* at coordinates U_1 and U_2, respectively. The objective is to locate these two sources using the data from a measurement performed at the coordinates K_1, or both K_1 and K_2. The correct combination of locations given by [1, 61] is obtained using one K coordinate, but there are several peaks of high correlation in other node combination numbers that are close to submultiples of U coordinates as e.g. the combination [1 21]. As expected, it does not happen when two K coordinates are used (Fig. 11 b).
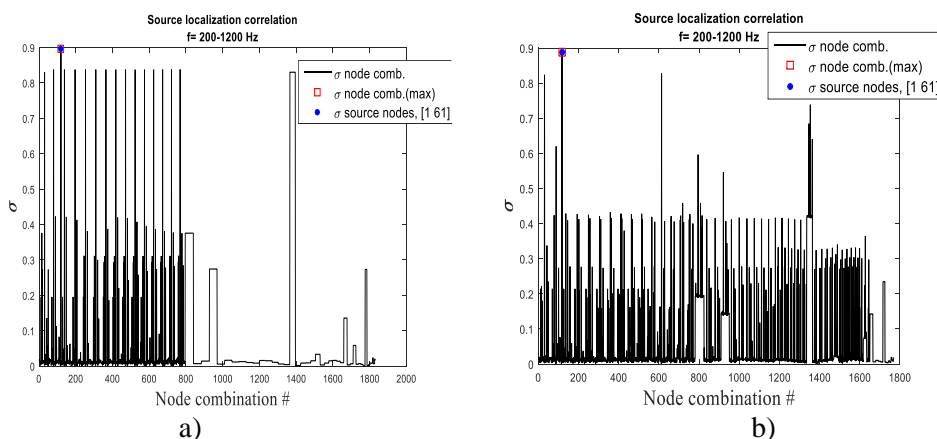

a)   b)

Figure 11 –Correlation results using: a) only coordinate *K1*; and b) both coordinates *K1* and *K2*.

9

5   Conclusions

A new concept of pressure transmissibility matrix in the frequency domain is proposed and applied to the acoustic source localization problem. The authors verified that: 1) obtained the correct positions of the sources acting; and 2) the search requires intensive computation.

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