ACUSTICA DE MEDIOS DEBILMENTE LINEALES

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ABSTRACT

We consider the propagation of acoustic waves in weakly nonlinear barotropic fluids when the density fluctuations are small by means of asymptotic methods for low Mach numbers and show that, depending on the coefficient that multiplies the nonlinear dependence of the pressure on the density, one can obtain Love's equations of elasticity, the equations for the propagation of sound in fluids containing bubbles, the nonlinear acoustics equations previously derived by Crighton, Lesser and Seebass, cnoidal waves, and topological solitons. Some numerical simulations that illustrate the propagation of acoustic waves in nonlinear barotropic fluids are presented.

RESUMEN

Se estudia la propagación de ondas acústicas débilmente no-lineales en fluidos perfectos barotrópicos cuando las variaciones de densidad son pequeñas por medio de métodos asintóticos a bajos números de Mach, y se muestra que, dependiendo del coeficiente que define la no-linealidad de la presión con la densidad, se obtienen las ecuaciones de Love de la teoría clásica de elasticidad, las de propagación del sonido en fluidos con burbujas y las ecuaciones de acústica no-lineal de Crighton, Lesser and Seebass, así como la existencia de ondas cnoidales y solitones topológicos.

INTRODUCTION

There is a vast literature on models of nonlinear acoustics, especially in one-dimensional domains and for weakly nonlinear media [1-4]. In such media, when nonlinearities are balanced by dispersion, solitary waves of permanent form that interact like particles, i.e., solitons, may appear, and completely integrable systems, e.g., the Korteweg-de Vries equation, characterized by an infinite number of conservation laws may occur. When the balance or governing equations contain dissipation terms, there may be only a finite number of conservation laws and dissipative solitons characterized by a balance amongst nonlinearities, dissipation, and dissipation may occur [5,6].

The purpose of this paper is several-fold. First, we present a one-dimensional model of weakly nonlinear compressible flows which is valid for gases and liquids and which is characterized by a barotropic approximation whereby the pressure is a quadratic function of the density. Second, by employing an asymptotic expansion for small Mach numbers, we derive a nonlinear wave equation for the velocity potential which contains a mixed fourth-order derivative and two quadratic nonlinearities. Third, we present a second-order accurate finite difference method in both space and time to solve the leading-order equation that results from the asymptotic analysis. Fourth, we solve numerically one-dimensional nonlinear dissipative and dispersive equations in order to illustrate the propagation of pressure pulses in one-dimensional barotropic media.

FORMULATION

In this paper, we consider a homoentropic flow of a lossless compressible fluid in one-dimensional space and in Eulerian coordinates. By neglecting the body force, assuming a unidireccional velocity field (u(x,t),0,0) where x and t denote the spatial coordinate and time, respectively, taking into account that the flow is irrotational and introducing the velocity potencial, one can write the continuity and linear momentum, i.e., Euler's, equations; the latter provides the fluid acceleration in terms of the pressure gradient for inviscid fluids. In this study, however, the inviscid Euler's equation contains a source term which depends on the fluid's density and velocity.

The two-equation system formed by the one-dimensional continuity and modified linear momentum equation is not closed, for it contains the density, pressure and velocity. However, for barotropic fluids characterized by a thermodynamic pressure that depends in a quadratic manner on the difference between the density and its equilibrium value, the system is closed. This quadratic dependence of the pressure on density includes both linear and nonlinear contributions; the linear contribution is akin to that found in isothermal flows of ideal gases, while the quadratic dependence on pressure results in a very interesting dynamical behaviour as shown in the next paragraphs.

By non-dimensionalizing the governing equations with respect to a characteristic length and a characteristic velocity, time with respect to the adiabatic acoustic time, i.e., the ratio of the length scale to the adiabatic sound speedy, and the density as the ratio of the density difference between the current density and the equilibrium value to the latter, it is an easy exercise to show that the resulting non-dimensional continuity and linear momentum equations contain the Mach number and a parameter that characterizes the nonlinear terms that have been introduced into the modified Euler's equation and that correspond to an averaged Euler's model of compressible flows in lossless fluids. Such a parameter is considered to be small in the weakly nonlinear formulation presented in this study.

By assuming that the nondimensional density difference is on the order of the Mach number which is, in turn, assumed to be also small and expanding the velocity potential in an asymptotic Poincaré expansion in terms of the Mach number, one can derive after some tedious and lengthy algebra, the following nonlinear equation

$$\Psi_{xx} - \Psi_{tt} + A\,\Psi_{xxtt} = \alpha\,[2\,(\theta - 1)\,\Psi_t\,\Psi_{xx} + ((\Psi_x)^2)_t], \qquad (1)$$

where $\Psi(x,t)$ is the velocity potential, α is the Mach number, $(\theta - 1)$ denotes the coefficient that multiplies the square of the density in the barotropic equation of state, and A is the nondimensional coefficient that multiplies the nonlinear terms introduced in the (modified) inviscid Euler equation. It must be noted that in the above equation $A = O(\alpha)$ and, therefore, terms of order α have been kept in the leading-order equation. If these terms are neglected, the above equation reduces to the well-known one-dimensional linear wave equation, i.e., $\Psi_{xx} - \Psi_{tt} = 0$. On the other hand, by taking the limit $\alpha \to 0$, the above equation reduces to Love's equation of classical elasticity theory and the inviscid van Wijngaarden's equation for sound waves in bubbly fluids [5]; in this case, A is related to the bubble radius. Moreover, the limit A →

0 yields the weakly nonlinear acoustic wave equation also known as the Crighton-Lesser-Seebass equation [7,8].

It must be pointed out that, in the derivation of the above equation, we have kept two regularization terms, i.e., the third term in the left and the term in the right hand side of the equation, which are on the order of the Mach number which was assumed to be small. These two terms have been retained in order to obtain a nonlinear wave equation, for their neglect would have resulted in the well-known one-dimensional wave equation which is neither dispersive nor nonlinear.

The above equation is invariant under mirror reflections in x, and translations in either t or x. The $x \to -x$ invariance implies that we only need consider right-travelling wave solutions, i.e., solutions characterized by $\Psi(x,t) = F(\zeta)$ where $\zeta = x - c\,t$ and c is the wave speed. Using this transformation, the leading-order equation for the velocity potential becomes a fourth-order nonlinear ordinary differential equation which can be integrated analytically once. By introducing $G = F'$ where the prime denotes differentiation with respect to ζ, a nonlinear second-order ordinary differential equation results. This equation has cnoidal wave solutions which are bounded periodic functions, travelling wave solutions of the sech-squared type, and kink or topological solitons of the hyperbolic tangent type [9].

NUMERICAL METHOD

The nonlinear leading-order wave equation presented in the previous section does not, in general, have analytical solutions except for the conditions mentioned in the last paragraph; therefore, its solution must be obtained numerically. In this paper, such an equation was solve as follows. First, the following dependent variable

$$\Phi = \Psi_t, \tag{2}$$

which allows to write the nonlinear wave equation (1) as

$$\Psi_{xx} - \Psi_{tt} + A\,\Psi_{xxtt} = \alpha\,[2\,(\theta - 1)\,\Phi\,\Psi_{xx} + \Psi_x\,\Phi_x], \tag{3}$$

Is introduced and then Equations (2) and (3) are linear in Φ and Ψ, respectively, i.e., if Φ were known, Ψ could be determined from Equation (3). Equations (2) and (3) are, however, coupled, and must, therefore, be solved in a coupled manner.

In this paper, Equations (2) and (3) were discretized by means of second-order accurate finite difference techniques in both space and time in infinite domains subject to radiation boundary conditions [10]. Because of the presence of the second-order derivatives and the use of three-time level finite difference equations, the resulting numerical method is not self-starting; the value of the velocity potential at the first time level was obtained by means of a (second-order accurate) Taylor series expansion in terms of $\Psi(x,0)$ and $\Psi_t(x,0)$, and the resulting system of nonlinear algebraic equations was linearized with respect to the previous time level, so that a system of linear algebraic equations was obtained. This system has a tridiagonal matrix which can be easily inverted by means of the Thomas algorithm.

RESULTS

The numerical method presented in the previous section was not only used to solve Equations (2) and (3) which correspond to a unidirectional lossless barotropic fluid for which some analytical solutions were described in the section on Formulation. It has also been applied to study acoustic wave propagation in dissipative and dispersive media.

Figure 1 illustrates the interaction of two pressure pulses in a dispersive and dissipative media. In the absence of dissipation, the two pressure pulses preserve their identity and behave as solitons upon interacting or colliding with each other. However, in the presence of dissipation, their amplitude decreases with time, their (x,t) trajectory is curved, and their width increases. The interaction of two pressure pulses in dissipative media such as the one illustrated in Figure 1 indicates that the curvature of the smaller pressure pulse is smaller than that of the larger one and that, their interaction is characterized by a local increase of amplitude. The results presented in Figure 1 also indicate that the amplitude of the two pressure pulses decreases after the interaction, although the two pressure pulses emanate from such an interaction as if they were particles moving in a dissipative media, and may, therefore, be referred to as dissipative solitons.

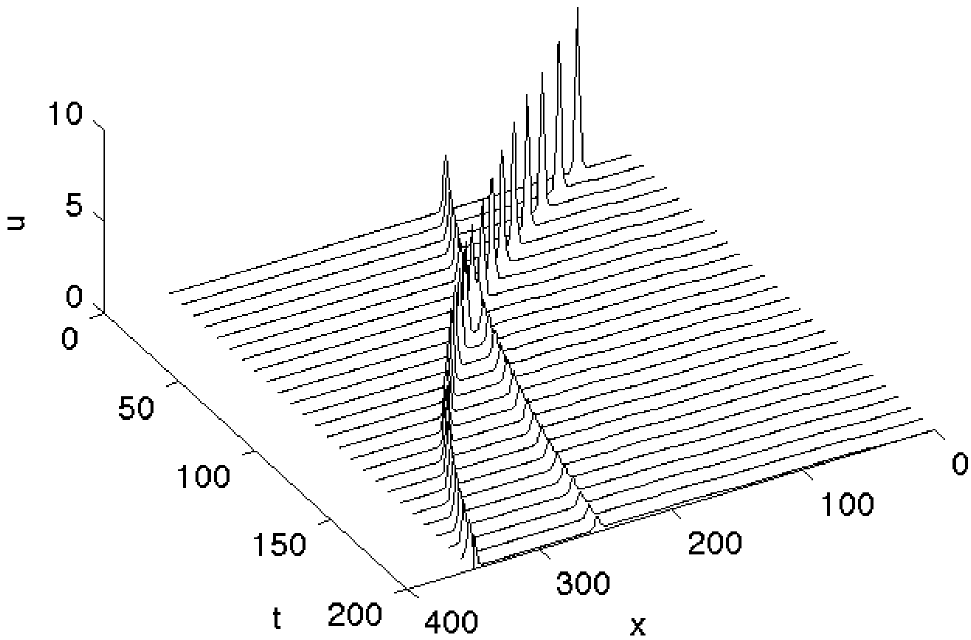

Figura 1. Interaction of two pressure pulses in a nonlinear dissipative and dispersive medium.

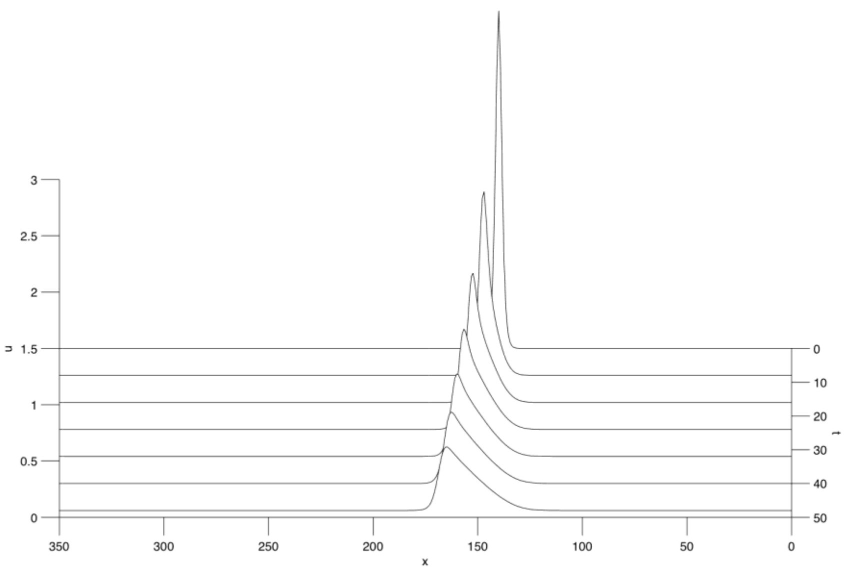

Figura 2. Propagation of a pressure pulse in a nonlinear dispersive media and front formation.

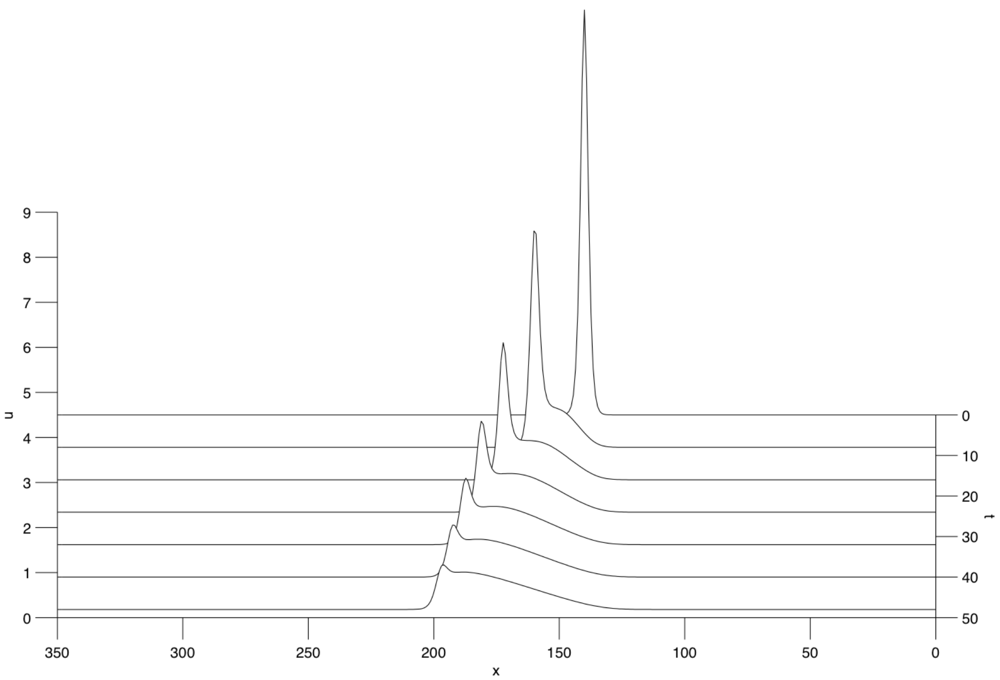

Figura 3. Propagation of a pressure pulse in a nonlinear dispersive media, front formation and tail dissipation.

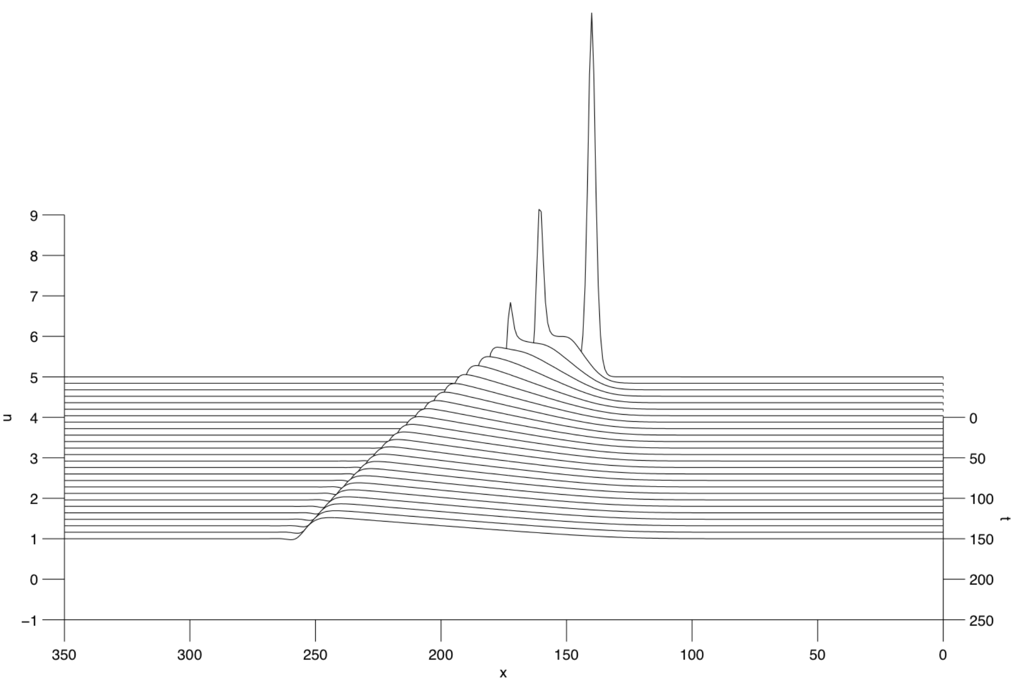

Figura 4. Propagation of a pressure pulse in a nonlinear dispersive media, front formation and tail dissipation.

Figure 2 illustrates the propagation and front steepening of a pressure pulse in a nonlinear dissipative media. Because of the dissipation, the amplitude and width of the pressure pulse increase, while the nonlinearities cause the steeping of the front. Pressure pulses of larger amplitude than the one illustrated in Figure 2 would result in a larger steepening at the pulse front, but no shock wave would result because of the finite viscosity/dissipation of the medium.

The results presented in Figure 3 clearly illustrate that, as the initial amplitude of the pressure pulse is increased, the front steeping may not grow as quickly as in Figure 2, but the pulse width may increase and result in a broader pulse whose amplitude exhibits an initial drop analogous to that of the initial pulse followed by a much smoother drop to a zero value downstream.

Figure 4 shows that, for large initial amplitudes of the pressure pulse, the initial propagation is similar to the one illustrated in Figure 3. However, at larger times, the steepening at the wave front which is enhanced by nonlinearities is damped away by dissipative effects, while the pressure pulse becomes wider as time increases.

The results presented in Figures 1-4 and others not presented here clearly show that the acoustic wave propagation in nonlinear media depends on the nonlinearities of the barotropic equation of state, the Mach number, and the nonlinearities, dispersion and dissipation. When the nonlinearities balance dispersion in a non-dissipative medium, soliton-type solutions may be found. With some abuse of language, it may also be stated that one may observe dissipative solitons in nonlinear dispersive and dissipative media under certain conditions. However, even in weakly nonlinear media, dispersion tries to steepen the wave front, while dissipation causes a decrease of amplitude and an increase of the width of the pressure pulse.

CONCLUSIONS

A one-dimensional model of barotropic fluids based on an averaged Euler lossless model that includes a nonlinear term in the inviscid Euler equation and employs a quadratic dependence of the thermodynamics pressure on the density has been analyzed asymptotically for small Mach numbers and small nonlinearities of the pressure dependence on the density. It has been shown that the leading-order equation in the Mach number is the well-known wave equation, but its regularization yields a nonlinear wave equation which includes Love's equation of elasticity, the nonlinear acoustics equations previously derived by Crighton, Lesser and Seebass, cnoidal waves, and topological solitons for different values of the two nondimensional parameters that appear in the nondimensional equation for the velocity potential .

A non-self-starting second-order accurate linearized finite difference method for the leading-order equation of the velocity potential has been proposed and used to determine the propagation of acoustic pulses in a nonlinear, dispersive and dissipative media.

It has been shown that dispersion causes a steepening of the pressure pulse, whereas dissipation decreases the pressure amplitude and increases the width of the pressure pulse.

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