VIBRATIONS OF SHALLOW SPHERICAL SHELLS AND GONGS: A COMPARATIVE STUDY

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ABSTRACT
Linear and nonlinear vibrations of shallow spherical shells with free edge are investigated experimentally and numerically and compared to previous studies on percussion instruments such as gongs and cymbals. The preliminary bases of a suitable analytical model are given. The prime objective of the work is to take advantage of the specific geometry of perfectly isotropic and homogeneous spherical shells in order to isolate the influence of curvature from other possible causes of nonlinearities. Hence, combination resonances due to quadratic nonlinearities are especially studied, for an harmonic forcing of the shell. Identification of excited modes is achieved through systematic comparisons between spatial numerical results obtained from a finite element modeling, and spectral informations derived from experiments.

INTRODUCTION
Vibrations of percussion instruments such as cymbals and gongs are essentially non-linear. This is due to the large deflections of the instrument, especially at the free edge, when struck by the blow of a mallet. Today, no theoretical model has been developed for these instruments and it is the aim of the present paper to contribute to filling this gap.

Our investigations started with an experimental study on cymbals subjected to an harmonic excitation [1]. While keeping the excitation frequency f_{exc} constant and increasing the amplitude, three distinct vibrational regimes were exhibited. The first one is periodic, defined by an harmonic spectrum governed by f_{exc}. A first bifurcation leads to the apparition of other spectral components, of frequencies that are incommensurate with f_{exc}, the resulting vibration being quasi-periodical. A second bifurcation leads to the third regime, which is chaotic. As a first attempt to write a physical model, a study was conducted on circular plates [2,3], which has led to a theoretical understanding of the periodic regime, validated by experiments. It was shown in particular that the problem can be modeled by a set of coupled second order nonlinear oscillators. The next step was then to study the quasi-periodical phase of the vibration.

Our second study of percussion instruments was conducted on a large orchestral gong (or Chinese tam-tam) [4]. The same route to chaos than for the cymbals was observed, and the quasi-periodical regime was especially investigated. Two results suggested us to think that quadratic non-linearities are to be considered in the model. The first result is that the flexural motion of the gong shows a clear distorsion due to the presence of a quadratic nonlinearity in

the oscillators. The second result is that the observed combination of resonances are governed by frequency relations of the form:

$$f_{exc} \approx f_n + f_m \qquad \text{or} \qquad f_{exc} \approx 2f_n \qquad (1)$$

where f_{exc} is the excitation frequency, f_m and f_n are eigenfrequencies of the gong. These frequency relations where also observed in the case of the cymbal [1], and are typical of quadratic nonlinearities [5]. These quadratic nonlinearities are a consequence of the curved geometry of the gong [6]. As a consequence, the flat plate model of [2] is inadequate since it exhibits cubic nonlinearities only. Our present objective is to investigate the nonlinear properties of a perfectly curved structure, of geometry as close as possible to those of gongs and cymbals, in order to facilitate the derivation of a theoretical model. The difficulties with real gongs are due to both their particular geometry and structure inhomogeneities [4]. The selected structure is a spherical cap. A number of caps, with different thickness, diameter and curvature, were especially designed in order to examine the influence of geometrical parameters on the quadratic non-linear effects. The caps are made of brass and are assumed to be perfectly homogeneous.

The next section is first devoted to an experimental linear modal analysis of the spherical caps. It enables to make several comparisons, first with circular plates, and then with the gong studied in [4]. In the nonlinear regime, several combination of resonances are reported. They are found to be similar to those observed in gongs and cymbals. In section 3, a theoretical model for non-linear transverse vibrations of a spherical cap with free edge is presented, in order to highlight the relevant parameters of the problem. The successive steps for the resolution of the problem are described.

EXPERIMENTS

The set-up is similar to the one used for the experimental study on gongs [4] and is extensively described in a more recent paper [3]. Only the main features are given here. The spherical shell is freely suspended by means of nylon threads (guitar strings). Is has been checked that the suspension does not alter significantly the values of eigenfrequencies and decay times. A small magnet is glued on the structure, and driven by a fixed coil.
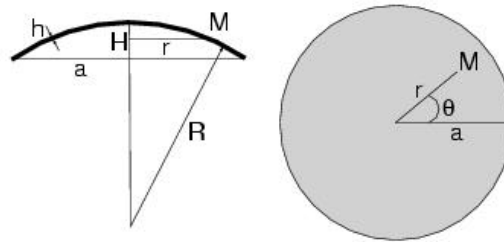
Modal analysis

Figure 1 : Geometry of the shell. Left: cross-sectional view. Right: top view.

Shell index	Radius of curvature R (mm)	Thickness h (mm)	$X = a^2/Rh$
1	925	1	98.4
2	1515	0.9	66
3	1515	1.5	39.6
4	4505	1	20

Table 1: Geometrical and elastic parameters of the shells. X is a nondimensional coefficient.

The geometries of the four shells used in our experiments are shown in Fig. 1 and Tab. 1. The eigenfrequencies of the shells are measured by exciting the structure with a filtered white noise. They are compared to those derived from a theoretical model of spherical shells with free-edge found in [7]. These eigenfrequencies are written:

$$\omega_{kn}^2 = \tilde{\omega}_{kn}^2 \left[1 + \frac{12\left(1-\nu^2\right)}{\mu_{kn}^4} X^2 \right] \qquad \text{with} \qquad X = \frac{a^2}{Rh} \qquad (2)$$

In Equation (2), a is the radius, R the radius of curvature and h the thickness are (Fig. 1). ω_{kn} denotes the eigenfrequency of the shell mode with k nodal diameters and n nodal circles, $\tilde{\omega}_{kn}$ denotes the corresponding eigenfrequency of the circular plate of same radius a. The coefficients μ_{kn} are the nondimensional numbers corresponding to the eigenvalues for a circular plate with free edge [8,2]. One can see in Equation (2) that the eigenfrequencies of the shells differ from the ones of the corresponding plate through the nondimensional geometrical parameter X only. In particular, Equation (2) gives the eigenfrequencies of a thin circular plate when X=0 ($R \rightarrow +\infty$). In the analytical procedure, the frequencies have been first calculated nondimensionally. The density is measured in a straightforward way. The Young's modulus corresponds to a standard value for brass. The value of the thickness h is adjusted so that the theoretical frequencies fit the experimental ones. The values are listed in Table 2. Table 3 gives the values of the coefficients μ_{kn} and the theoretical and measured shell eigenfrequencies ω_{kn}, for shell number 2 referenced in Table 1.

Material	Young's mod. E (N/m^2)	Density ρ (kg/m^3)	Radius a (mm)	Poisson's coefficient ν
Brass	85 10^9	7.97 10^3	300	0.3

Table 2: Measured parameters of the shell.

Mode index	Circular plate coefficient μ_{kn}	Theoretical freq. (Hz)	Measured freq. (Hz)
(2,0)	5.25	8.9	?
(3,0)	12.20	21.4	?
(4,0)	21.6	38.1	35.3 and 35.6
(5,0)	32.28	58.7	57.6 and 58.4
(6,0)	46.2	82.9	83.2 and 83.9
(7,0)	62.0	110.5	110.8 and 111.9
(8,0)	81.0	141.1	141.3 and 142.0
(9,0)	102.5	211.5	208.5 and 213.4
(10,0)	126.5	251.5	244.5 and 250.1
(11,0)	153	293.8	292.2 and 295.1
(12,0)	182	339.5	334.4 and 338.3
(0,1)	9.1	342.1	**224**
(0,2)	38.6	348.3	354

Table 3: Theoretical and measured eigenfrequencies of spherical shell Nr 2 (X=a^2/Rh=66). The first index refers to the number of nodal radii, while the second indicates the number of nodal circles.

In contrast to the case of gongs, Table 3 shows that the frequencies of a relatively large number of asymmetric modes are smaller than the lowest axisymmetric mode (0,1). This property is a consequence of essentially two facts:
a) The edge of the shell is free, and thus its transverse motion is significantly less constrained than in the case of a gong with a stiff ring [4].
b) The geometry of the shell is curved, which yields more stiffness than for the gong, especially in its central part. This property is also responsible for the fact that asymmetric modes with at least one nodal circle ($k \neq 0$, $n \geq 1$) have eigenfrequencies lower than $f_{0,1}$.

The asymmetric modes are grouped here by pairs, for which both configurations have almost identical eigenfrequencies. The corresponding modal shapes have the same pattern, with a quadrature spatial phase shift (see Fig. 2). This special feature comes from the fact that in a

perfect structure of revolution, each asymmetric eigenfrequency degenerates into two independent modes [9,2]. The slight differences between the measured frequencies of both configurations in the case of our spherical caps (see Table 3), are due to small imperfections, mainly induced by the nylon threads. They appear to be negligible with respect, for example, to the differences measured in the case of our gong [4]. This can be explained by the fact that the gong shows structural and material inhomogeneities due to hammering and thermic treatment during its making [4]. Taking into account the existence of two configurations for the asymmetric modes is essential, as these configurations are independently taking part in the combination of resonances which are usually observed [4,2].

Table 3 shows an almost perfect agreement between calculated and measured eigenfrequencies except for the lowest axisymmetric mode. This effect might be due to the presence of the magnet (0.5 g) glued at the center of the shell. However, more systematic experiments and calculations are needed here in order to validate this assumption.
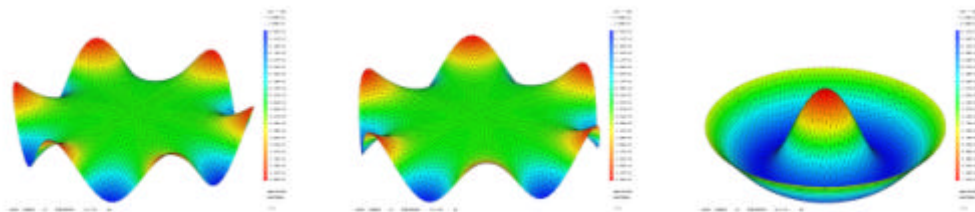

Figure 2. From left to right: the two configurations of the (6,0) asymmetric mode, and modal shape of the axisymmetric (0,2) mode for the spherical cap.

Combination of resonances in the non-linear regime.

Forcing frequency (Hz)	combination of resonances (Hz)	subharmonics (Hz) (particular case)
224 (0,1)		112 (7,0)
354 (0,2)	318 (11,0) shifted+ 35 (4,0)	
	247 (10,0)+ 107 (7,0) shifted	
	211 (9,0)+ 143 (8,0)	
444 (0,3)		222 (0,1)
	336 (12,0) + 108 (7,0) shifted	

Table 4: Summary of the most prominent experimentally observed combination of resonances.

Table 4 shows some examples of observed combination of resonances, for the cap Nr 2 (see Table 1) excited at its center with an harmonic force of frequency successively adjusted to one particular axisymmetric eigenfrequency. All combinations of resonances in Table 4 correspond to the two cases presented in Equation (1). Fig 3 shows the velocity spectrum of the cap excited at 354 Hz (f_{02}). These phenomena are similar to those observed in cymbals [1] and gongs [4].
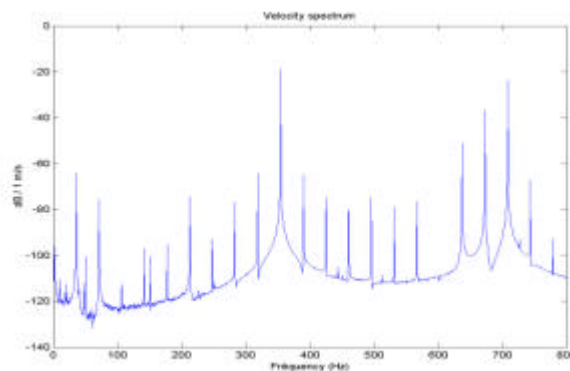

Figure 3: Velocity spectrum of one selected point on the cap. Fexc=348 Hz. The resonances excited through quadratic nonlinearities (see Table 4) are clearly visible.

THEORY

Governing equations

The nonlinear partial differential equations in polar coordinates for the spherical cap are the following [10]:

$$D\Delta^2 w - L(w,F) + \frac{1}{R}\Delta F + \rho h\,\ddot{w} = p(r,\theta,t) \tag{3}$$

$$\Delta^2 F + \frac{Eh}{2}L(w,w) - \frac{Eh}{R}\Delta w = 0 \tag{4}$$

$$L(w,F) = w_{,rr}\left(\frac{F_{,r}}{r} + \frac{F_{,\theta\theta}}{r^2}\right) + F_{,rr}\left(\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2}\right) - 2\left(\frac{w_{,r\theta}}{r} - \frac{w_{,\theta}}{r^2}\right)\left(\frac{F_{,r\theta}}{r} - \frac{F_{,\theta}}{r^2}\right) \tag{5}$$

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{6}$$

In Equations (3)-(6), w refers to the vertical transverse displacement, F is the Airy stress function and $p(r,\theta,t)$ is the external pressure. D is the flexural rigidity. The underlying assumption are that w is of the order of the shell thickness, and that the shell is shallow (h<<a<<R). Letting $R \to +\infty$ in Eqs (3)-(4) yields the nonlinear equations for a nonlinear plate [2]. The case $L = 0$ corresponds to the linear spherical shallow shell model. Both conditions together lead to the well-known linear plate equations. The curvature factors (proportional to 1/R in Eqs. (3) and (4)) are responsible for a linear coupling between the transverse displacement w and the longitudinal stretching of the shell elements (which is a function of F). Equation (5) shows that $L(w,w)$ is a quadratic function of the displacement w. As a consequence, the Airy function F is both linear and quadratic in w through (3). Similarly, $\Delta F/R$ is a quadratic function of the displacement in (2), whereas $L(w,F)$ has quadratic and cubic terms in w. This important feature illustrates the fact that both cubic and quadratic terms necessary coexist in the nonlinear spherical shell equations, in contrast to the nonlinear plate equations which contain cubic terms only. To summarize this discussion, one can say that the quadratic nonlinearity is due to the coexistence between the nonlinear function L and the linear curvature terms.

Modal projection and resolution.

The solution w is expanded onto the linear eigenmodes of the shell:

$$w(r,\theta,t) = \sum_{p=1}^{+\infty}\Phi_p(r,\theta).q_p(t) \tag{7}$$

where Φ_p is the pth modal shape. With little mathematics [11], one can show that the time functions q_p are solutions of a infinite set of nonlinear second-order differential equations, coupled by quadratic and cubic terms:

$$\ddot{q}_p + \alpha_p\dot{q}_p + \omega_p^2 q_p = -\sum_{u=1}^{+\infty}\sum_{v=1}^{+\infty}\alpha_{uvp}q_u q_v - \sum_{s=1}^{+\infty}\sum_{u=1}^{+\infty}\sum_{v=1}^{\infty}\tilde{A}_{suvp}q_s q_u q_v + Q_p \tag{8}$$

In Equation (8), α_p and Q_p denote the damping coefficient and the forcing of the pth mode, respectively. Coefficients α_{uvp} and $\tilde{A}_{suvp}$ are functions of the modal shapes Φ_p.

The next step consists of truncating the infinite set of Equation (8) by assuming that the forced vibration of the shell is governed by a finite number of modes only [5]. The quasi-periodic regime corresponding to a subharmonic resonance between mode (0,1) and mode (7,0), for example, is governed by three oscillators: the first oscillator corresponds to mode (0,1) (of frequency $f_{01} = 224Hz$), and the two other oscillators correspond to the two configurations of

mode (7,0) (of frequencies $f_{70}^1 = 110.8\,\text{Hz}$ and $f_{70}^2 = 111.9\,\text{Hz}$, respectively), The vibration shown in Figure 3 can be described by at least a set of five oscillators, sine at least one axisymmetric and two asymmetric modes are nonlinearly coupled. The resolution of the resulting problems for a finite number of oscillators can be either performed by approximate analytical methods such as the multiple scale method [5], or by a direct numerical resolution (using a Runge-Kutta algorithm, for example). This work is currently in progress.

CONCLUSION

So far, our study on the large magnitude forced oscillations of thin shallow spherical shells with free edge can be summarized as follows:

Like for the gong, combination resonances dues to quadratic nonlinearity are present. These resonances are governed by frequency laws, such as Eq. (1). For thickness, radius and diameter comparable to gongs, the rank of the lowest axisymmetrical modes is significantly higher for a spherical shell. This property is due to the free edge of the shell, compared to the relatively stiff edge of gongs, and to the more pronounced curvature of the cap. In practice, this means that a significant number of combination resonances can be found here for the (0,1) and (0,2) mode, whereas one had to excite at frequencies at least equal to the (0,3) mode of the gong in order to obtain a large number of combinations. For spherical shells with free edge, the lowest frequencies are essentially asymmetric (n,0) modes. The homogeneity of the shell facilitates the analysis in the sense that each asymmetric mode is characterized by a doublet of frequencies with close values. This analysis is confirmed by numerical experiments. In addition to the interest of comparing spherical shells with gongs, in the context of musical acoustics, our measurements contribute to yield experimental and numerical data on shells with free edge, for which there are very few published papers, as far as we are aware.

The nonlinear phenomena encountered in gongs and cymbals, and more generally in thin structures, are due to geometrical nonlinearities, and directly linked to the large amplitude of the vibrations (of the order of the thickness, or even larger. In the particular case of spherical shells, it has been shown in Equation (8) that the curvature is responsible for the presence of the quadratic terms. From the present comparison between nonlinear behavior of shells and gongs, which shows high similarity in the nonlinear quasiperiodical regime, one can reasonably claim that the imperfections observed in gongs have relatively no effect on the nonlinear behavior of these instruments compared to the curvature.

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