NUMERICAL MODELING OF THE NONLINEAR PROPAGATION OF TRANSIENT ACOUSTIC SIGNALS

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Campos-Pozuelo C.[1]; Vanhille C.[2]

[1] Instituto de Acustica
Serrano, 144, 28006 Madrid, Spain
Tel: +34.91.561.88.06; Fax: +34.91.411.76.51; E-mail: ccampos@ia.cetef.csic.es
[2] ESCET. Universidad Rey Juan Carlos.
Tulipán, s/n. 28933 Móstoles, Madrid. Spain
Tel: +34.91.664.74.82; Fax: +34.91.664.74.90; E-mail: c.vanhille@escet.urjc.es

ABSTRACT. A numerical study of high-amplitude transient signals propagating in an absorbing, homogeneous fluid is presented. The work is motivated in many applications where high intensity waves, which can not be described by linear laws, are involved (biomedical research, high power ultrasonics, etc.). Differential equations are written in lagrangian coordinates, and the full nonlinear equation is solved by means of a finite difference algorithm. Calculations are performed exclusively in the time domain, giving all the harmonic amplitudes by only one resolution step, and allowing the analysis of the evolution of the waveform for any original signal: gaussian, rectangular pulses, periodic excitation, etc. Numerical results are presented for waveform distortion and shock formation for plane transient and harmonic waves. Spatial and initial pulse shape dependences are specially analysed.

INTRODUCTION

This paper deals with a numerical study of high-amplitude acoustic signals propagating in an absorbing, homogeneous fluid. Many applications exist where high intensity waves, which can not be described by linear laws, are needed. In particular, the development of medical ultrasound as a therapy tool (hyperthermy, lithotripsy, etc.) involves high amplitudes and, thus, nonlinear propagation. In the other hand we observe an increase in the amplitudes of the ultrasonic fields applied in diagnostic in order to have greater penetration and greater resolution [1]. The theory accompanying this development can not be made anymore under the assumption of linear propagation. In this framework, the purpose was to develop a numerical method for studying the nonlinear propagation of plane waves by using the full nonlinear equation derived in lagrangian coordinates. Some authors have used eulerian coordinates and the "retarded time" variable associated with the propagation in the $+x$ direction: $\boldsymbol{t} = t - x/c_0$, which allows them to reduce by one the order of the differential equation for wave motion [1,2]. In this paper we solve the full nonlinear differential equation written in lagragian coordinates by using natural spatial and time coordinates. This implies the need of imposing a non-reflecting boundary condition. The formulation is written in the time domain, allowing the analysis of the evolution of the waveform for any original signal: periodic excitation, gaussian, rectangular pulses, etc. In addition, all the harmonic components are obtained by only one resolution step, with the consequent save in computation time.

The equations of the acoustical problem are presented in Section I. Section II presents the numerical algorithm. The numerical scheme is experimented and results are presented, validated and commented in Section III. The conclusions of the paper are finally given.

I. FUNDAMENTAL EQUATIONS

Nonlinear waves propagating in a homogeneous thermoviscous fluid are studied. The Tait-Kirkwood equation of state for isentropic fluids has been considered [1]. The one-dimensional full nonlinear wave equation written in lagrangian coordinates is considered [3]:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \chi\left(p_0 + \pi\right)\frac{1}{\left(1+\frac{\partial u}{\partial x}\right)^{\chi+1}}\frac{\partial^2 u}{\partial x^2} + \rho_0 \nu \beta \frac{\partial^3 u}{\partial t \partial x^2}\,, \qquad (1)$$

where p_0 is the ambient pressure, ρ_0 the initial state density, π and χ are characteristic constants of the fluid, u is the displacement, v is the kinematic shear viscosity, and b is the so-called viscosity number. t and x are, respectively, the time and one-dimensional spatial coordinates.

No approximations have been made about the acoustic Mach number value or about the attenuation parameter, i.e., the only limitations on pressure amplitude in the model are those derived from the isentropic approximation [3]. However, since we consider the propagation of a (transient) wave within an unbounded domain, even in the case of very high acoustic Mach number, the isentropic property of the fluid can not be questioned.

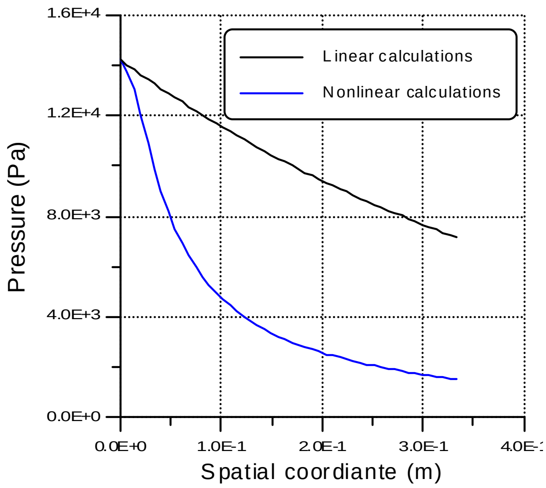

Fig. 1. Comparison of the amplitude evolution at the fundamental frequency for a strongly nonlinear progresive wave ($f = 20$ kHz $\alpha = 2.06\ \mathrm{m}^{-1}$)

Boundary conditions

We consider progressive plane waves and a source placed at $x = 0$. Then, the following boundary conditions are written:

$$x = 0 \quad u(0,t) = f(t) \qquad (2.a)$$

$$x = L \quad c_0 \frac{\partial u}{\partial x}(L,t) = -\frac{\partial u}{\partial t}(L,t), \qquad (2.b)$$

where c_0 is the small-amplitudes value of the sound speed, L is the length of the domain considered in calculations. The "quasilinear" non-reflecting condition (2.b) will be tested by comparing the numerical results with analytical results obtained for an harmonic wave. $f(t)$ is the source function, i.e., the excitation of the medium defined as a function of time. The method supports any source function. In this article some numerical experiments are made by considering different dependencies for $f(t)$: a continues wave and two gaussian pulses of different length.

The fluid is assumed to be initially at complete rest: particle displacement and velocity are null at t = 0. The following initial conditions are then employed:

$$t = 0 \quad \begin{cases} u(x,0) = 0 \\ \dfrac{\partial u(x,0)}{\partial t} = 0 \ \forall x \neq 0 \end{cases}. \qquad (3)$$

Heating

It is well known that the absorption of sound leads to heat generation in the acoustic medium. Moreover, temperature is a very important parameter in applications, especially in medical applications. When dealing with very high amplitude waves the nonlinear attenuation is completely dominant (see Figure 1) and then the heating predicted by a linear theory is not correct anymore and it has to be calculated in the new nonlinear framework. We calculate the heating rate and the temperature increase by using the isentropic hypotheses [3] and the Fourier's law,

$$\kappa \frac{\partial^2 \theta}{\partial x^2} + \left(\mu_B + \frac{2}{3}\mu \right)\left(\frac{\partial^2 u}{\partial t \partial x} \right)^2 = 0, \tag{4}$$

$$q = -\kappa \frac{\partial \theta}{\partial x}, \tag{5}$$

where q is the heat-flux, θ the absolute temperature, κ is the coefficient of thermal conductivity, and μ and μ_B are the viscosity and bulk viscosity. No-additional hypotheses about the acoustic Mach number has been made. These equations are solved by using a conventional finite differences algorithm [4].

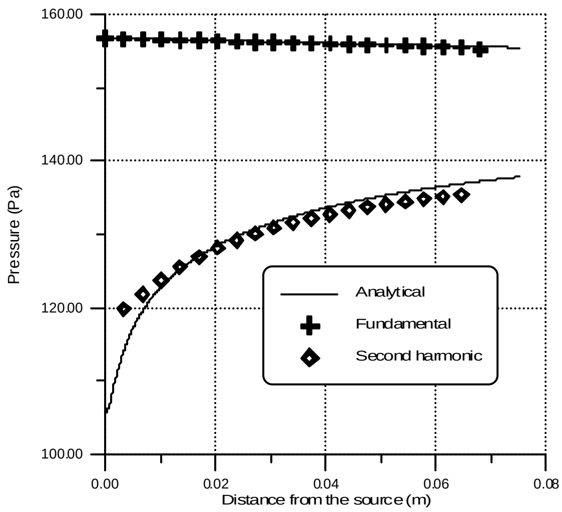

Fig. 2. Comparison with analytic results for $f = 20\,\mathrm{kHz}$, 5 periods, 5 wavelengths, and $\alpha = 2.06\,\mathrm{m}^{-1}$

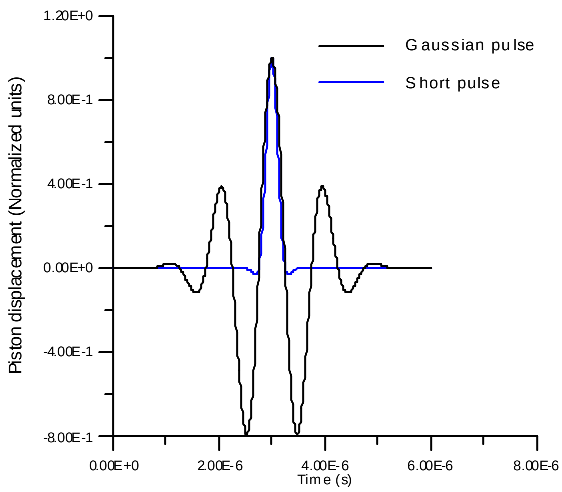

Fig. 3. Excitations signals

II. NUMERICAL FORMULATION

In this section the numerical formulation developed for solving the acoustical problem is described. It is based on the finite-difference method [5] and has been included in the numerical code Snow-ac [3,8].
The numerical scheme is created by considering two dimensionless independent variables: $X=x/\lambda$ and $T=\omega t$, where λ is the wavelength of the signal and ω is its angular frequency, and an uniform discretization of the X-T space. The differential equations of the previous section are numerically treated as in reference [3,6]. Displacements are evaluated at every point of the discretized space. The method is implicit and conditionally convergent [3,6]. Boundary conditions (2) are taken into account at each time-step. A notable feature of the method is the linear structure of the system of equations obtained at each time-step for solving the nonlinear problem.

The mechanical perturbations of the wave reach the wavelength number i at the period number i, and thus many values of the displacement are null. This phenomenon is taken into account in the algorithm. These null values are not evaluated. This fact generates an important saving of operations, storage and CPU time. For this purpose computations are first led on the perturbated zone during the first group of periods only, by considering the initial conditions. The number of periods of the group depends on the kind of pulsed excitation. The second phase computes the values of the displacement on the perturbed zone during the second group of periods, by considering a continuity condition with the first part. And as so on up to the last group of periods of the problem.

At each time-step of the j-th period the system of equations is solved by means of an economic and fast method based on a LU decomposition valid for the whole j-th period [7].

Pressure and heating rate are calculated from the displacement values by means of classic finite-difference schemes.

III. VALIDATION AND RESULTS

A validation of the numerical method is presented by comparing with analytical results referred to a quasi-linear case. The analytical model is based in a perturbation technique in the frequency domain. We assume a solution consisting in the addition of two terms in the form $u = u_l + u_2$, $p - p_0 = p_l + p_2$ and with $u_2 << u_l$ and $p_2 << p_l$, where u_l (p_l) represents the first-order solution of Eq. (1) and u_2 (p_2) the second-order correction. The additional assumption of small attenuation is made and dissipation is taken into account by using a complex wave number. With these approximations, the following analytical solution is obtained:

$$
\begin{aligned}
u_l &= u_0 e^{j(wt-kx)} & u_2 &= -\frac{g+1}{8} k^2 u_0{}^2 x e^{2j(wt-kx)} \\
p_l &= j r_0 c_0^2 k e^{j(wt-kx)} & p_2 &= -r_0 c_0^2 \frac{g+1}{8} k^2 u_0{}^2 (1 + j2kx) e^{2j(wt-kx)}
\end{aligned} \qquad (6)
$$

where $k = k_0 - ja$, and $k_0 = \frac{w}{c_0}$. In Figure 2 analytical and numerical results are compared. A harmonic source of amplitude $u_0 = 25\,m\mathrm{m}$ is considered at the frequency $f = 20000\,\mathrm{Hz}$. $c_0 = 340\,\mathrm{m/s}$, $g = 1.4$ and $r_0 = 1.29\,\mathrm{kg/m}^3$. We consider $a = 2.06\,\mathrm{m}^{-1}$. $h = 0.01$ is employed. The numerical sound pressure distribution of the first and second harmonics at the last instant of the study is shown to coincide with the analytical one. The harmonic decomposition of the numerical signal is obtained by means of a FFT. These good results validate the numerical method presented.

Some results are now presented referring to the propagation of transient signals. The source function is written as $f(t) = u_0 e^{-[x_b(t-t_0)]^2} \cos(wt)$. We have chosen two short signals: a very short pulse $\left(x_B = 5\times10^6\right)$ and a gaussian pulse $\left(x_B = 10^6\right)$. The evolutions of the waveshapes and shock formation are studied and the importance of the initial waveshape analysed. In all cases we have considered a fluid with acoustic properties similar to tissues of the body (with the exceptions of lung, bone, and fat): $c_0 = 1500\,\mathrm{m/s}$, $g = 6.2$ and $r_0 = 1000\,\mathrm{kg/m}^3$, $a = 11\,\mathrm{m}^{-1}$ [1]. We consider a frequency of 1 MHz, quite usual in medical applications. In Figure 3 we see the considered displacement at the piston in normalized units. The displacement amplitude at the piston is $u_0 = 1.5\,mm$, which means an initial pressure amplitude of the order of 15 MPa, quite typical in medical applications, both diagnostics (in focalized region) and therapy. In Figure 4 we show the evolution of the waveshape for the two considered cases. We observe that the strong harmonic distortion occurs at the first wavelengths from the source in both cases. When distance to the source increases, the central frequency of the short gaussian pulse decreases. At 7.2 cm from the source the pulse has an amplitude of the order of 27% of its initial value and its frequency is about three times less than the original; no much harmonic distortion affects this state of the pulse. Thus, for this type of excitation signal, the main effect involved is the nonlinear attenuation associated with the harmonic distortion in a nonlinear medium with a dispersion relation of the type ω^2 for the absorption. When dealing with a wider gaussian pulse, the strong distortion affects again at the first wavelengths from the source, at 7.2 cm from the source the amplitude of the wave is of the order of 32% of its original amplitude.

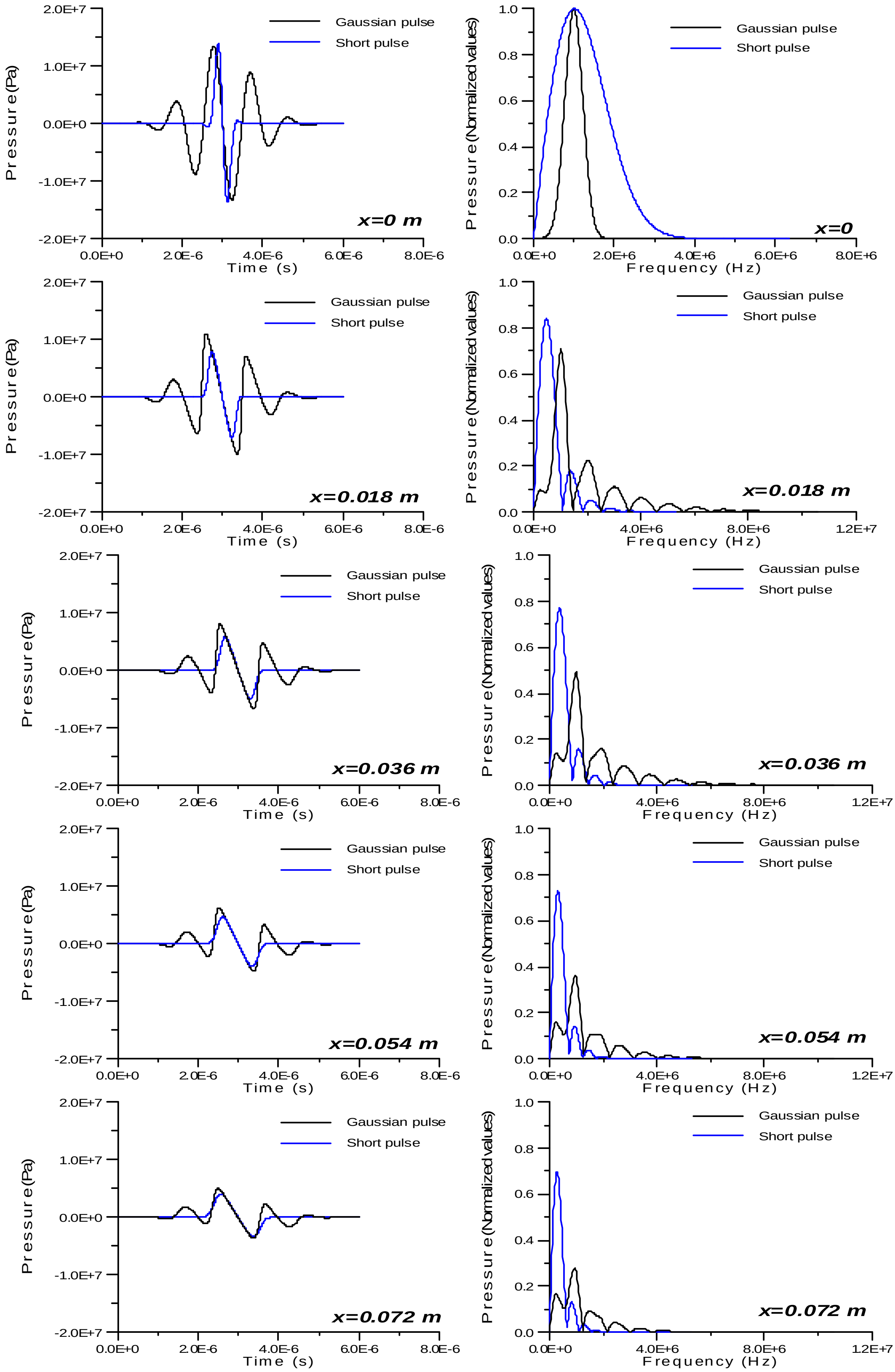

Fig. 4. Waveshapes and Fourier decomposition of the two analyzed signals at different distances from the source

The central frequency of the fundamental has decreased only in 12.5% in front of the 300% of the short pulse. This can be interpreted because the low frequencies, which are quite less attenuated, when the dispersion relation considered is ω^2, are more present in the short pulse. Another important difference is the apparition for this kind of signals of the low frequency. This low frequency increases fast with the distance to the source, and corresponds to the ***self-demodulation*** of the initial signal. In this case, this frequency is $0.22 \times f$, being *f* the central frequency of the excitation, which corresponds to the modulation frequency of the initial pulse. In figure 5 we show the evolution of the amplitude at the fundamental and at the demodulation frequency with the distance.

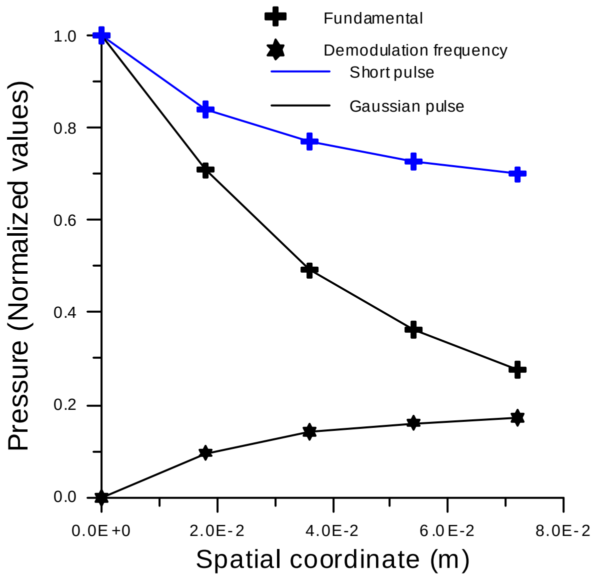

Fig. 5. Pressure amplitude at the fundamental and at the demodulation frequency

CONCLUSIONS

An study of the nonlinear propagation of high amplitude waves has been presented. The analysis is based in a finite-difference algorithm which solves the full nonlinear wave equation written in Lagrangian coordinates. Bulk attenuation has been considered (a w^2 dispersion relation) and no approximations have been made about the absorption parameter value. The algorithm works in the time domain. This means that the whole wave-shape is obtained by only one resolution. The algorithm has been validated by comparison to a "quasi-linear" analytical solution. The method has been applied to the analysis of the propagation of transient signals. We have shown the strong dependence of the observed nonlinear effects on the initial frequency content of the signal. The analysis of the results showed the importance of nonlinear effects when considered the propagation of relatively high amplitude transient signals.

REFERENCES

1. *Nonlinear Acoustics*, ed. By Hamilton MH and Blackstock DT. Academic Press: New York, 1998
2. D. T. Blackstock, "Propagation of plane sound waves of finite amplitude in nondissipative fluids", J. Acoust. Soc. Am., 34, 9-30 (1962)
3. Vanhille C and Campos-Pozuelo C. "Numerical model for nonlinear standing acoustic waves and weak shocks in thermoviscous fluids", J. Acoust. Soc. Am. 2001; **109**: 2660-2667
4. Vanhille C and Campos-Pozuelo C., "Three computational models for quasi-standing nonlinear acoustic waves", submitted for publication to Journal of Computational Acoustics
5. Smith GD. *Numerical solution of partial differential equations, Finite Difference Methods*. Clarendon Press: Oxford, 1985.
6. Vanhille C and Campos-Pozuelo C. "A High-Order Finite-Difference Algorithm for the Analysis of Standing acoustic Waves of Finite but Moderate Amplitude", J. Comput. Phys. 2000; **165**: 334-353.
7. Vanhille C and Campos-Pozuelo C. "Étude numérique dondes ultrasonores progressives d'amplitude finie", *Actes du 6ème Congrès Français d'Acoustique*, (2002).