SPECTRAL WAVEGUIDE FINITE ELEMENTS

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ABSTRACT
A spectral element method is described which enables reduced wave equation problems defined in regular, arbitrary length regions to be solved as a set of coupled problems over neighbouring domains. A combination of trial functions are considered, namely the specific eigenfunctions of a differential operator and a set of hierarchical polynomials. The coefficients in the representation of the acoustic pressure are obtained by imposing continuity across element interfaces and including the given boundary conditions and sources. Accurate approximations are obtained for computations of local perturbations of the ambient fluid and a study of an optimal non-reflecting boundary is discussed.

INTRODUCTION

A variational formulation for the two-dimensional Helmholtz equation in an acoustic waveguide is presented. The objective is to calculate sound transmission through local perturbations located either within the fluid or at the fluid boundary. The case considered is that of wave propagation from a monofrequency point source in a waveguide that is assumed to be straight and arbitrarily long with uniform rectangular cross-section. Characterisation of an absorbing boundary is modelled by a locally reacting impedance condition. Parameters governing the fluid wavespeed are also allowed to vary along the longitudinal direction of the waveguide. The proposed approach can provide accurate solutions over domains of arbitrary length in a certain direction; namely, an infinite waveguide solution, a semi-infinite waveguide solution and a solution within a finite rectangular region.

The problems studied here are specific to waveguides whose fluid domains only vary in width. For general domains there has been a significant effort in improving the numerical modelling capability for linear steady-state acoustic problems in two and three dimensions for interior or exterior acoustic domains. Numerical methods for solving reduced wave differential equations can be categorized according to their local or global nature. The most general local approaches are the finite element method (FEM) and the boundary element method (BEM). These discrete methods rely on spatial discretizations that are small compared to the wavelength of the problem. However, these are methods well suited to general wave propagation problems. For finite and boundary element methods the underlying approximating functions are usually polynomials with *compact support* in the acoustic region. For high wavenumbers this choice

leads to systems of equations with a large number of degrees of freedom. Whereas, for acoustic waveguide problems a judicious choice of basis functions, which have an underlying global nature, may lead to significantly less number of degrees of freedom. For the new spectral element method presented here the basis functions inherently possess some of the wave nature of the acoustic field. Consequently, for acoustic problems posed for a waveguide geometry the spectral methods require considerably less degrees of freedom than the discrete methods.

The spectral finite element method (SFEM) applied to waveguide problems, referenced in [1] – [3] can be viewed as a merger of the dynamic stiffness method and the finite element method. Specifically the method is based on a variational formulation for non-conservative motion in the frequency domain. The SFEM has been used to study vibration in beam frame--works [1], beam--stiffened railway cars [2] and for fluid-filled pipes [3]. Finnveden used the spectral pipe elements for various cases from assessment of approximate theory, [4], to experimental SEA calculations, [4]. Use of a variational formulation for the spectral method provides a natural basis for approximations and a simple tool for combination with standard finite elements. In the following spectral finite elements are combined to solve a waveguide problem, and in particular a possible *optimal* local boundary condition is presented that approximates the true non-local boundary condition at an interface for a half-space. Optimal local boundary conditions have been discussed comprehensively by Givoli in [5] and [6], the ideas used here differ in that a reflection coefficient is minimised for *almost all* angles for all plane-wave incidence.

FINITE ELEMENT SOLUTION

In acoustic waveguides with constant cross--section the solutions of the equations of motion, {\it i.e.} Helmholtz equation, are exponential terms describing wave propagation in an axial direction with corresponding cross--sectional modes. Polynomials are derived using a procedure developed by Finnveden in [3] to describe fluid motion in the cross--section of pipe structures. The hierarchical polynomials appear in the trial functions used in the approximation for acoustic pressure,

$$p_N(x,z) = \boldsymbol{g}(z)\,\boldsymbol{B}\,\hat{\boldsymbol{p}}(x) \qquad (1.1)$$
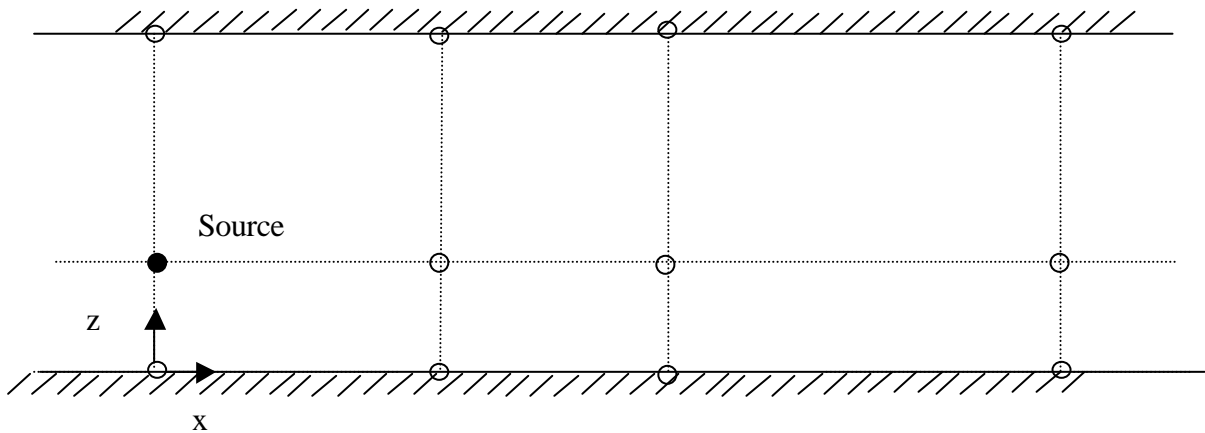
where $g_j(z) = z^{j-1}$ is the jth entry in the vector $\boldsymbol{g}(z), 0 \le z \le h$. The polynomials are non-dimensionalised so that the variable $\boldsymbol{x} = z/h$ appears in the definitions of the approximating functions.

Inserting the approximating functions into a variational formulation for the Helmholtz equation, see [7] for details , evaluating certain derivatives and integrals in the z-direction, one obtains an expression for the dynamic stiffness matrices :

$$\mathbf{K}_2 = -r\mathbf{B}^T \int_0^h \mathbf{g}(z)^T \mathbf{g}(z)\,dz\,\mathbf{B} \tag{1.2}$$

$$\mathbf{K}_0 = r\,k^2\mathbf{B}^T \int_0^h \mathbf{g}(z)^T \mathbf{g}(z)\,dz\,\mathbf{B} - r\mathbf{B}^T \int_0^h \frac{d\mathbf{g}(z)^T}{dz}\frac{d\mathbf{g}(z)}{dz}\,dz\,\mathbf{B} +$$
$$ikr\,\mathbf{B}^T\left(z_h\mathbf{g}(h)^T\mathbf{g}(h) - z_0\mathbf{g}(0)^T\mathbf{g}(0)\right)\mathbf{B} \tag{1.3}$$

assuming an absorbing condition at the upper and lower boundaries. After a little matrix algebra a system, ($N \times N$), of second order differential equations for the unknown function $\overset{\cup}{p}$ follows

$$\mathbf{K}_2\frac{\partial^2 \overset{\cup}{p}}{\partial x^2} + \mathbf{K}_0\,\overset{\cup}{p} = 0 \tag{1.4}$$

Since the system of equations (1.4) has constant matrix coefficients the solution may be written in the form

$$\overset{\cup}{p}(x) = \mathbf{X}e^{i\lambda x} \tag{1.5}$$

where X is a vector representing wave amplitudes and λ wavenumbers determined by a suitable matrix eigenvalue problem. Using the definition for propagating waves in (1.5) and polynomials in the z-direction it is possible to form an appropriate trial or wave influence function over a finite axial length - $L \leq x \leq L$.

Non-reflecting boundary condition

The application of the spectral finite element method on unbounded exterior domains involves a domain decomposition of the exterior region. On an entirely artificial boundary at $z = 0$, it is possible to prescribe artificial boundary conditions (ABC) that approximate the exact non-local boundary condition, [5] and [6].

Consider a plane wave P_{inc} incident on the boundary $z = 0$ at angle θ from the normal. Writing the addition of the incoming and scattered waves as

$$P = P_{inc} + P_{ref} = e^{ik(z\cos q + x\sin q)} + R_M e^{ik(-z\cos q + x\sin q)} \tag{1.6}$$

it is possible to associate the reflection coefficient with a generalised M^{th} local boundary condition at $z = 0$ as

$$\frac{\partial P}{\partial z} = \sum_{i,j=0} a_{i,j}\frac{\partial^M P}{\partial x^i\partial z^j}. \tag{1.7}$$

It is also possible to recover the reflection coefficient R_0 from the zero-order or rc boundary condition, ($a_{0,0} = ik$), and after a little manipulation the well-known expression for the reflection coefficient is obtained,

$$R_0 = \frac{\cos q - 1}{\cos q + 1}. \tag{1.8}$$

Now, reformulating the generalized boundary condition (1.7) into a form in which the general reflection coefficient may be established:

$$R_M(q) = \frac{\cos q + a_{2,0}\sin^2 q + a_{4,0}\sin^4 q + a_{6,0}\sin^6 q + \ldots - 1}{\cos q + a_{2,0}\sin^2 q + a_{4,0}\sin^4 q + a_{6,0}\sin^6 q + \ldots + 1}. \qquad (1.9)$$

Minimizing $R_M(q)$ under a suitable norm it is possible to find rational values for the unknown coefficients $a_{i,j}$. In turn the resulting coefficients in equation (1.7) may be incorporated into the finite element model (1.2) and (1.3). The following two examples give some sense of the scope of the method for a waveguide with varying density and a simple exterior problem.

RESULTS

Density variation

In this example the Helmholtz equation is solved for an infinite waveguide with dimension $-0.5 \pounds z \pounds 0.5m$ including a rigid boundary where the density of the fluid is allowed to vary from ambient state $r = 1.21 kg m^{-3}$ over the region, $9 \pounds x \pounds 10m$ to the right of the source. Absorbent material also covers the upper and lower boundaries of the inhomogeneous region and the source is located at the lower boundary $(0,-0.5)$ at fixed frequency excitation 220 Hz. Figure 2 shows the absolute pressure along the waveguide just below the centre line at z = -0.3. The left hand figures (a)-(c) show pressure variation for a total rigid waveguide and the right hand figures, (d)-(f) show variation in density with absorbing material.
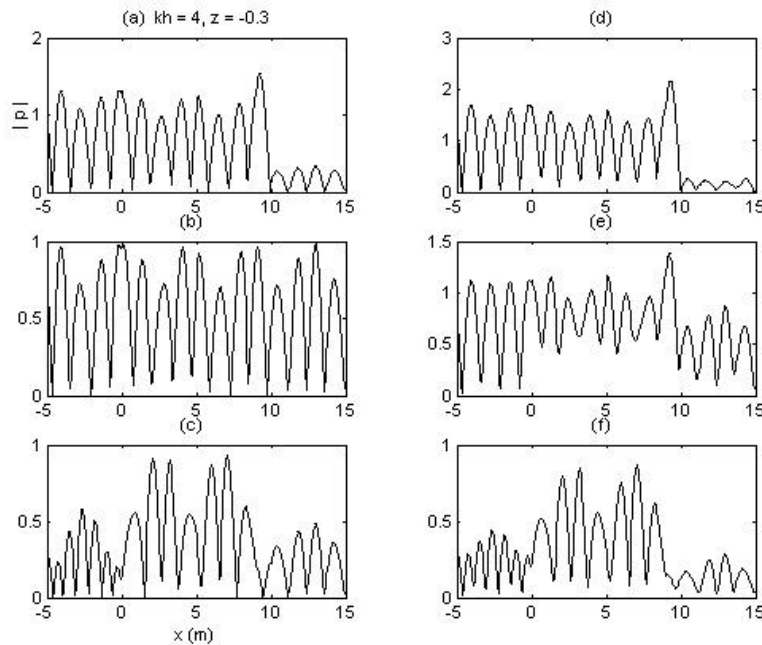

Figure 2. Rigid waveguide (a) ñ $= 0.21$ kg m^{-3} (b) ñ $= 1.21$ kg m^{-3} (c) ñ $= 10.21$ kg m^{-3} and absorbing waveguide $z = 1/100$ (d) ñ $= 0.21$ kg m^{-3} (e) ñ $= 1.21$ kg m^{-3} (f) ñ $= 10.21$ kg m^{-3}

<u>An optimal boundary condition</u>

As a simple example to validate the high order, local,non-reflecting boundary condition above a point source was located at a rigid boundary x=0m and z = -4m in an acoustic half-space. By the method of images the exact solution is a sum of two Hankel functions of the first kind of zero order. For the numerical solution two waveguide layers were assembled with a depth of 2m each. Furthermore three coefficients, including the zero order coefficient

$$a_{0,0} = ik, \, a_{2,0} = \frac{1}{2}, a_{4,0} = -\frac{1}{8}, \tag{1.10}$$

were incorporated into the spectral finite element solution via equation (1.3). Figure 3 shows good agreement between the analytic and numerical solutions across the waveguide domain for non-dimensional wavenumber $kh = 17$ at x=3m.
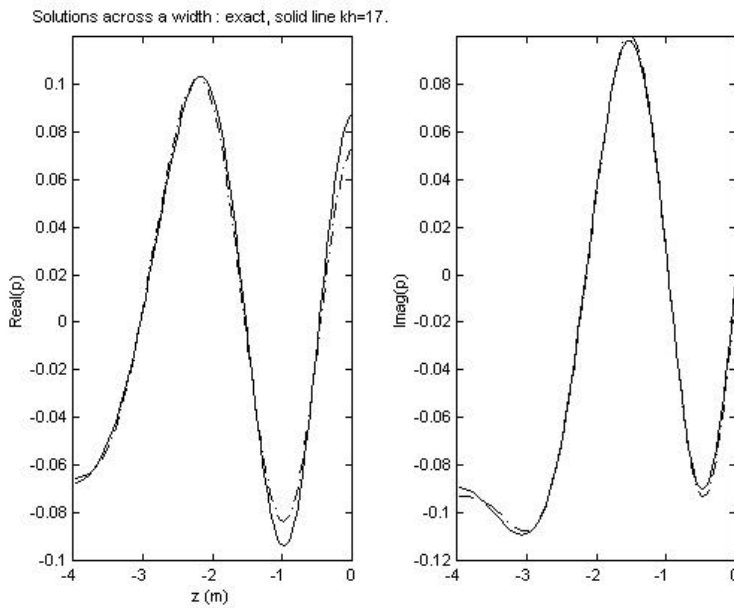

Figure 3. Real and imaginary parts of finite element solution (dash-dot) to acoustic half-space problem versus the analytic solution (solid).

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