EFFICIENT ITERATIVE SOLUTION OF THE HELMHOLTZ EQUATION WITH AN OPTIMIZED PML

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ABSTRACT
We consider efficient iterative solution of the three-dimensional Helmholtz equation with a perfectly matched layer (PML). Certain parameters of the PML are first optimized to minimize the reflection by the layer. We use an algebraic fictitious domain method with a separable preconditioner for the iterative solution of the linear system arising from the finite element discretization. Such methods are based on embedding the domain into a larger one with a simple geometry, which allows preconditioning with fast direct solvers. We use a parallel implementation of the method to solve high-frequency acoustic scattering problems with impedance-type boundary conditions.

1 INTRODUCTION

The scattering of time-harmonic acoustic waves by a 3D obstacle can be modelled by an exterior boundary value problem for the Helmholtz equation. The finite element solution of such a problem requires the unbounded domain to be truncated at a finite distance, and it becomes necessary to approximate the radiation condition at the truncation boundary. There are several different techniques to reduce spurious reflections caused by this artificial boundary such as local absorbing boundary conditions, the exact nonlocal boundary condition, and infinite elements. The idea of the perfectly matched layer (PML) approach is to bound the domain by a finite layer of absorbing medium instead of a boundary. It is possible to construct a layer in which the magnitude of an incident wave of any frequency or angle of incidence is decreased exponentially with distance into the layer, thereby leading to low reflection. This method was first proposed by Berenger for the two-dimensional time-dependent Maxwell equations [2].

In this article, we consider the numerical solution of the Helmholtz equation with a rectangular PML surrounding the computational domain. We use algebraic fictitious domain methods, which are based on the idea of embedding the original domain into a larger domain with simple geometrical form. The linear system resulting from the discretization is replaced by an equivalent, but enlarged, system corresponding to the simple-shaped domain. In this way, it is possible to use fast direct solution methods as efficient preconditioners for the enlarged system. We incorporate the fast direct solver for the Helmholtz equation with a PML described in [6] into the fictitious domain method considered in [4] and in [5]. Furthermore, we determine the parameters for the PML by using the optimization method suggested in [7].

We use the finite element discretization on locally perturbed orthogonal meshes, called locally fitted meshes. With such discretization and fictitious domain preconditioning, we are able to realize the GMRES method in a subspace of dimension $O\left(N^{2/3}\right)$ where N is the dimension of the total system. Thus, it is possible to use the partial solution method to solve the linear systems with the preconditioner [1][10].

We developed efficient parallel implementation of the iterative solution in [5]. Our implementation is based on MPI communication and can be used in various parallel computers. We present results of numerical experiments in an SGI Origin 2000 parallel computer.

2 FORMULATION AND DISCRETIZATION

2.1 Helmholtz Equation with a PML

The scattering problem is illustrated in Figure 1, where the obstacle is denoted by D and the exterior domain is truncated by a rectangular layer of thickness δ. We consider the following variable-coefficient Helmholtz equation in the rectangular domain $\Pi = \prod_{i=1}^{3}\left(\breve{x}_i, \hat{x}_i\right)$:

$$
\begin{aligned}
-\nabla \cdot \left(A \nabla u(x)\right) - \omega^2 \gamma u(x) = 0, \quad & x \in \Pi / \overline{D}, \\
\Phi(x,\omega) u(x) = g(x,\omega), \quad & x \in \partial D, \\
u(x) = 0, \quad & x \in \partial \Pi.
\end{aligned}
$$

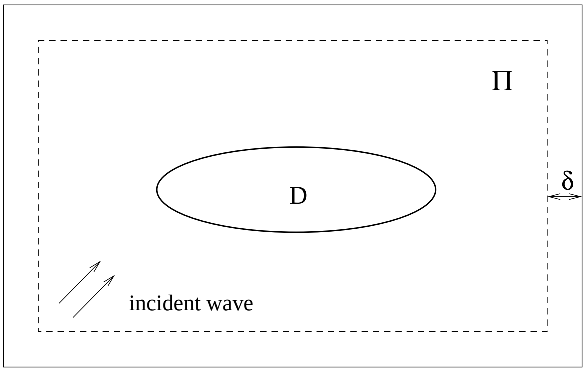


Figure 1: Illustration of the scattering problem with an ellipse.

The function g corresponds to a plane incident wave propagating in the direction of the wave vector. On the boundary of the obstacle, we have an impedance boundary condition defined by

$$
\Phi(x,\omega) = \frac{\partial}{\partial n} - \nu.
$$

The diagonal matrix A is of the form

$$
A(x) = \operatorname*{diag}_{k=1,2,3}\left\{\frac{\gamma(x)}{\gamma_k^2(x)}\right\}, \quad \gamma(x) = \prod_{k=1}^{3} \gamma_k(x_k),
$$

where the functions γ_k are given by

$$
\gamma_k(x_k) = 1 + \frac{i\sigma_k(x_k)}{\omega}, \quad \sigma_k(x_k) = \begin{cases} \sigma_0 \left(\dfrac{\breve{x}_k + \delta - x_k}{\delta}\right)^p, & x_k < \breve{x}_k + \delta, \\ 0, \quad \breve{x}_k + \delta \le x_k \le \hat{x}_k - \delta, & \\ \sigma_0 \left(\dfrac{x_k + \delta - \hat{x}_k}{\delta}\right)^p, & x_k > \hat{x}_k - \delta, \end{cases}
$$

where $\sigma_0, \delta,$ and p are nonnegative constants.

For the weak formulation of the equation, we introduce the function space

$$V = \left\{ v \in H^1\left(\Pi / \overline{D}\right) : v\big|_{\partial\Pi} = 0 \right\}$$

and the sesquilinear form

$$a(u,v) = \int_{\Pi/\overline{D}} \left(A\nabla u \cdot \nabla\overline{v} - \omega^2 \gamma\, u\overline{v}\right) dx - \int_{\partial D} \nu\, u\overline{v}\, ds.$$

Then, the weak formulation can be represented in the following form: Find $u \in V$ such that

$$a(u,v) = f(v) \equiv \int_{\partial D} g(x,\omega)\overline{v}\, ds, \quad \forall\, v \in V.$$

2.2 Finite Element Discretization

We discretize the variational problem by using the finite element method with piecewise linear elements and mass lumping. The computational domain is partitioned into tetrahedrons using a special procedure, which results in a locally perturbed orthogonal mesh called a locally fitted mesh. The use of such meshes is motivated by the fact that they allow us to realize the iterative solution procedure in a low-dimensional subspace. For details we refer to [5].

We denote the finite element space corresponding to V by V_h. The discrete problem is then the following: Find $u_h \in V_h$ such that $a(u_h, v_h) = f(v_h), \forall\, v_h \in V_h$. It can be shown that this problem is uniquely solvable for sufficiently small h provided the corresponding continuous problem is uniquely solvable. The approximation of oscillatory functions by functions in V_h can be shown to be quasioptimal beyond the minimum resolution limit $\omega h = \pi/2$ [8]. However, quasioptimal error estimates for the finite element solution have been obtained only with the assumption that the quantity $\omega^2 h$ is sufficiently small. In addition to the interpolation error, the error in the Galerkin finite element approximation involves also a pollution term of order $\omega^3 h^2$.

The discretization leads to a complex symmetric linear system $\mathrm{A}u = f$. We divide the nodes of the locally fitted mesh into three groups: The group γ contains all the nodes on the boundary of D, while the group $\gamma-$ consists of the nodes having a common element edge with a node in γ. The group I contains the rest of the nodes. By reordering the degrees of freedom according to this partitioning, the linear system can be written in the block form

$$\begin{pmatrix} \hat{\mathrm{A}}_{II} & \hat{\mathrm{A}}_{I\gamma-} & 0 \\ \hat{\mathrm{A}}_{\gamma-I} & \mathrm{A}_{\gamma-\gamma-} & \mathrm{A}_{\gamma-\gamma} \\ 0 & \mathrm{A}_{\gamma\gamma-} & \mathrm{A}_{\gamma\gamma} \end{pmatrix} \begin{pmatrix} u_I \\ u_{\gamma-} \\ u_\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_\gamma \end{pmatrix}.$$

3 FICTITIOUS DOMAIN METHOD

3.1 Preconditioner for the Enlarged Problem

We replace the original linear system by an enlarged system of the form

$$\hat{\mathrm{A}}\hat{u} = \begin{pmatrix} \mathrm{A} & \mathrm{A}_u \\ 0 & \mathrm{A}_d \end{pmatrix} \begin{pmatrix} u \\ u_d \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where $\ker \mathrm{A}_d \subseteq \ker \mathrm{A}_\mathrm{u}$. We construct the blocks A_u and A_d by using the method described in [4].

The fictitious domain preconditioner for the enlarged system corresponds to the problem

$$-\nabla \cdot \left(A \nabla u(x)\right) - \omega^2 \gamma u(x) = 0, \quad x \in \Pi,$$
$$u(x) = 0, \quad x \in \partial\Pi,$$
which is discretized using an orthogonal tetrahedral partitioning of the domain Π. This mesh coincides with the locally fitted mesh outside an h-layer neigboring the boundary ∂D, and we end up with a separable matrix B [6]. By an appropriate renumbering of the nodes, we are able to represent this matrix in a block form, which corresponds to the block representation of the matrix A.

3.2 Iterative Method in a Subspace

We use the preconditioned GMRES method to solve the enlarged system. On each iteration, it is necessary to solve one linear system with the preconditioner and multiply the result with the system matrix. These two tasks may be significantly optimized by taking into account that only the rows corresponding to the node groups γ and $\gamma-$ are different in $\hat{\mathrm{A}}$ and B. For this reason, only $O\left(N^{2/3}\right)$ of the rows of the matrix $C \equiv \hat{\mathrm{A}} - B$ are nonzero. We have the identity $\hat{\mathrm{A}} B^{-1} w = \left(C B^{-1} + I\right) w$, which implies that if the initial approximation is the right-hand side of the enlarged system then all iteration vectors will belong to the subspace $\operatorname{im} C$. In other words, it is necessary to store only those components which correspond to the node groups γ and $\gamma-$. This fact allows us to apply the partial solution method to compute the matrix-vector multiplications of the form $C B^{-1} w$.

3.3 Partial Solution Method

The partial solution method for the direct solution of linear systems corresponding to elliptic equations in rectangular domains was introduced by Banegas in [1] and by Kuznetsov and Matsokin in [10]. The method is a special implementation of the classical method of separation of variables, and its efficiency is based on the assumptions that only a sparse set of the solution components of the linear system is required and that the right-hand side vector has only a few nonzero components. Then, the partial solution procedure is obtained directly from the method of separation variables by neglecting arithmetical operations with the zero components.

4 OPTIMIZATION OF THE PML

We studied in [6] and in [7] the optimization of the parameters σ_0 and p to reduce the reflection by the PML (see also [3]). As suggested by the results in [7] we choose $p = 3.5$ and optimize σ_0 to minimize the modulus of the reflection coefficient. In our tests, the thickness of the layer is kept fixed. For the problem considered here, the explicit form of the reflection coefficient is given in [6].

5 NUMERICAL EXPERIMENTS

We used a simple model problem in which the scatterer is the ellipsoid given by
$$\frac{x^2}{10^2} + \frac{y^2}{1.75^2} + \frac{z^2}{1.75^2} = 1.$$
A rectangular outer boundary is well-suited with such scatterers of high aspect ratio. We solved the Helmholtz problem of Section 2.1 with two different wave numbers such that the minimum distance between the boundaries $\partial D \text{ and } \partial\Pi$ was $\lambda = \omega / 2\pi$ and the thickness of the PML

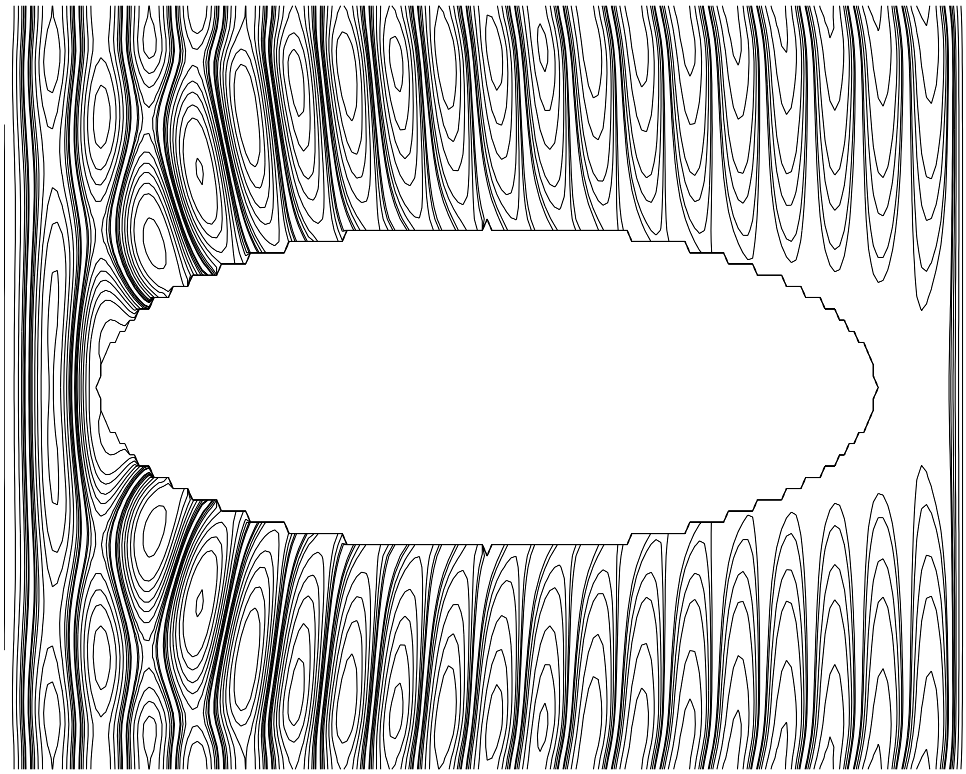
was $\delta = \lambda/2$. The coefficient ν in the impedance boundary condition was $-\omega$. Our results with two different discretization resolutions λ/h are collected in the following table:

ω	λ/h	N	$\dim(\operatorname{im} C)$	Iter	time
$2\pi/5$	10	5.1e4	2.5e3	129	11.4 s
$2\pi/5$	20	3.9e5	1.0e4	116	62.1 s
$4\pi/5$	10	1.2e5	1.0e4	606	259 s
$4\pi/5$	20	9.6e5	4.1e4	545	1570 s

In the table, iter gives the number of GMRES iterations to reach convergence, and column time is the wall clock time of the solution. The results were computed using eight processors.

We see that the method seems to be robust with respect to the discretization resolution, but unfortunately the number of iterations grows rapidly when the frequency increases, and we were unable to solve the problems with $\omega = 8\pi/5$ due to limited computer resources. We conclude that the efficiency of the fictitious domain method we have used is not satisfactory with impedance-type boundary conditions. With Dirichlet boundary conditions this method works rather well as shown by the results in [5].

The real part of the total wave (incident wave + scattered wave) on the xz-plane for the case $\omega = 4\pi/5$ is illustrated in the following figure. The contour plot includes the solution also in the PML, where the scattered wave decays exponentially.



6 ACKNOWLEDGEMENTS

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