Preconditioners for Acoustic Problems: AILU

Martin J. Gander *

Abstract

AILU (Analytic ILU) is an incomplete LU (ILU) preconditioner which is based on the underlying physical problem. Numerical experiments have shown that AILU is much more effective than classical ILU. We derive for the Helmholtz equation as a model problem for acoustics the optimal parameters to be used in two types of AILU preconditioners and analyze their performance.

1 Introduction

Discretizing an elliptic problem $\mathcal{L}(u) = f$ with a finite element, finite difference or finite volume method, one obtains a matrix equation $A\boldsymbol{u} = \boldsymbol{f}$ where A is large and sparse. Hence Krylov methods are the methods of choice to solve these problems, but the methods can have serious convergence problems, for acoustic problems see [5]. These problem remain when matrix based, black box preconditioners are used. Even a preconditioner which comes close to a factorization, like ILU(1e-2), can not alleviate the situation [5].

We analyze in this paper two approximate factorizations based on the underlying continuous operator. These factorizations were introduced in [5] and analyzed in depth for symmetric positive definite problems in [4]. They are based on the analytic factorization of the elliptic operator, which has been of interest for some time, see for example [3, 9]. The first use of this approach in an iterative fashion to solve a large system of linear equations was proposed by Nataf in [7] and extended by Nataf, Loheac and Schatzman in [8]. The idea was also used by Giladi and Keller to solve a convection dominated convection diffusion equation arising in an asymptotic analysis in [6]. However the performance of these factorizations was not satisfactory, because they were missing a link between the analytic factorization and the exact block LU decomposition. This link was established in [4] and leads to approximate factorizations of high quality. We show in this paper for acoustic problems why this link is necessary for good performance. The factorizations discussed in this paper are related to work at the fully discrete level by Wittum in [12, 13] extended later by Wagner [10, 11] and Buzdin [2] for positive definite problems. A recursive application for 3d problems can be found in [1].

2 The Continuous Analytic Factorization

Given an elliptic operator $\mathcal{L}(u)$ we write the operator as a product of two parabolic operators,

$$\mathcal{L}(u) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2)(u) \tag{2.1}$$

where Λ_1 and Λ_2 are positive operators up to a compact operator. The first factor represents a parabolic operator acting in the positive x direction and the second one a parabolic operator acting in the negative x direction. We focus in the sequel on the Helmholtz operator $\mathcal{L} = (-\omega^2 - \Delta)$ as a model for acoustic problems.

Taking a Fourier transform of $\mathcal{L}$ in y we obtain

$$\mathcal{F}_y(-\omega^2 - \Delta) = -\partial_{xx} + k^2 - \omega^2 = -(\partial_x + \sqrt{k^2 - \omega^2})(\partial_x - \sqrt{k^2 - \omega^2}) \tag{2.2}$$

and thus we have the continuous analytic factorization

$$(-\omega^2 - \Delta) = -(\partial_x + \Lambda_1)(\partial_x - \Lambda_2) \tag{2.3}$$

*Dept. of Mathematics and Statistics, McGill University, Montreal, mgander@math.mcgill.ca

where $\Lambda_1 = \Lambda_2 = \mathcal{F}_y^{-1}(\sqrt{k^2-\omega^2})$. Note that the Λ_i are non local operators in y because of the square root. This non-locality corresponds to the fill in in an ILU preconditioner, because an ILU also produces an operator looking forward, namely the L, and another operator looking backward, namely the U. For a precise identification see [4]. Hence a local approximation of the nonlocal symbol in the factorization leads to a new type of preconditioner we call AILU (Analytic Incomplete LU).

Suppose we approximate the nonlocal symbols Λ_i by a local approximation $\Lambda_i^{app} = \mathcal{F}_y^{-1}(p+qk^2)$, $p, q \in \mathbb{C}$, $\Re(q) > 0$. Then the approximate new operator is $\mathcal{L}^{app} = \mathcal{F}_y^{-1}(-\partial_x^2 + p^2 + 2pqk^2 + q^2k^4)$ and an iterative method to solve the original problem would then only solve two local parabolic problems at each step, $\mathcal{L}^{app}(u^{n+1}) = (\mathcal{L}^{app} - \mathcal{L})(u^n) + f$. Hence the parameters $p, q \in \mathbb{C}$ should be chosen so that the convergence rate $\rho := \mathcal{F}(1 - (\mathcal{L}^{app})^{-1}\mathcal{L})$ is as small as possible except for a few frequencies which will be taken into account once the stationary iterative solver is replaced by a Krylov method. In Fourier, the convergence rate becomes after some calculations

$$\rho = \frac{(p+qk^2)^2 + w^2 - k^2}{(p+qk^2)^2 + k_x^2}, \tag{2.4}$$

where we have also transformed the x direction with Fourier parameter k_x.

To choose $p, q \in \mathbb{C}$, we first note that we can restrict our analysis to $k \geq 0$, because ρ depends on k^2 only. Since the free parameters are complex, we have four real parameters to use in the optimization process. To simplify the approach, we choose the parameter p such that the convergence rate vanishes at $k = 0$, $p := i\omega$, and the parameter q such that the convergence rate vanishes again at an intermediate frequency $\overline{k} > \omega$, $q := \frac{(\overline{k}^2-\omega^2)^{1/2} - i\omega}{\overline{k}^2}$. Estimating the lowest frequency in the x-direction by zero, $k_x = 0$, the modulus of the convergence rate becomes

$$R(k, \overline{k}, \omega) := |\rho|^2 = \frac{k^4(k-\overline{k})^2(k+\overline{k})^2}{(k^4 - 2\omega^2k^2 + \overline{k}^2\omega^2)^2}. \tag{2.5}$$

Theorem 1 *The convergence rate $R(k, \overline{k}, \omega)$ is bounded by one for all k if and only if $\overline{k} = \sqrt{2}\omega$.*

Proof First we note that at $k = \omega$ the convergence rate $R = 1$, independently of what we choose for the optimization parameter $\overline{k}$. The derivative of R with respect to k is

$$\frac{\partial R}{\partial k} = 4\frac{k^3(\overline{k}-k)(\overline{k}+k)(\overline{k}^4\omega^2 - \overline{k}^2k^4 - 2\overline{k}^2\omega^2k^2 + 2\omega^2k^4)}{(k^4 + \omega^2\overline{k}^2 - 2\omega^2k^2)^3}, \tag{2.6}$$

which evaluated at $k = \omega$ gives $4\frac{\overline{k}^2 - 2\omega^2}{\omega(\overline{k}^2-\omega^2)}$. Hence, if $\overline{k} \neq \sqrt{2}\omega$, then the slope at $k = \omega$ is non-zero, but at $k = \omega$ we have $R = 1$ which implies that $R > 1$ for some values of k and shows the only if part. For the if part, we find from (2.6) at $\overline{k} = \sqrt{2}\omega$ that the extrema are at 0, ω and $\sqrt{2}\omega$. Since $R = 0$ at $k = 0$ and $k = \sqrt{2}\omega$, these are minima, and in between at $k = \omega$ we find the only maximum, where $R = 1$. Hence $R \leq 1$ except maybe for large k. But $\lim_{k\to\infty} R(k, \overline{k}, w) = 1$ which shows that $R \leq 1$ for all k if $\overline{k} = \sqrt{2}\omega$. ■

At first sight the optimal choice of $\overline{k}$ seems to be determined by Theorem 1, because with the choice $\overline{k} = \sqrt{2}\omega$ the method converges for all frequencies except for $k = \omega$ and $k = \infty$. Using Krylov acceleration, these two modes would be taken care of by the Krylov method and the preconditioned method would converge overall. But in a numerical computation two additional issues come into play: first a numerical grid can not carry arbitrary large frequencies, there is a maximum frequency $k_{\max}$ which can be estimated for a grid-size h by $k_{\max} = \pi/h$. Second the domain is bounded, which leads to a discrete spectrum, $k = dn$, where n is an integer and d is some spacing. Hence we do not need $R \leq 1$ for all k, we only need to consider $k \in K := (0, \omega - \delta\omega) \cup (\omega + \delta\omega, k_{\max})$ where $\delta\omega$ is a parameter we can choose to determine how many of the modes around $k = \omega$ should be taken care of by the Krylov method. For example choosing $\delta\omega = d/2$, there is at most one mode left for the Krylov method, if we manage to obtain $R < 1$ for $k \in K$. We first obtain an important corollary of Theorem 1 in this case.

Corollary 2 *With fixed optimization parameter $\overline{k} = \sqrt{2}\omega$ the asymptotic convergence rate for $k \in K$ with $k_{\max} = \pi/h$ as $h \longrightarrow 0$ is*

$$R(k_{\max}, \overline{k}, \omega) = 1 - 4\frac{\omega^4}{\pi^4}h^4 + O(h^6). \tag{2.7}$$

Proof The result follows by simply expanding $R(\pi/h, \sqrt{2}\omega, \omega)$ for h small. ∎

Now the question arises if there is a better choice of the parameter $\overline{k}$ if $k \in K$ only. This leads to the min-max problem

$$\min_{\omega+\delta\omega<\overline{k}<k_{\max}} \left(\max_{k\in K} R(k, \overline{k}, \omega) \right), \quad K := (0, \omega - \delta\omega) \cup (\omega + \delta\omega, k_{\max}). \tag{2.8}$$

Theorem 3 *If*

$$\begin{aligned} k_{\max} &> \sqrt{\frac{\delta\omega^4 + 4\omega^3\delta\omega + \omega^4 - 2\omega^2\delta\omega^2 + \sqrt{\omega^8 - 2\omega^4\delta\omega^4 + \delta\omega^8 + 16\omega^6\delta\omega^2}}{2(\delta\omega(2\omega - \delta\omega))}} \\ &= \frac{\omega^{3/2}}{\sqrt{2\delta\omega}} + \frac{5\sqrt{2\omega}}{8}\sqrt{\delta\omega} + O((\delta\omega)^{3/2}), \end{aligned} \tag{2.9}$$

then the solution of the min-max problem (2.8) is given by

$$\overline{k} = \frac{\sqrt{2}(\omega + \delta\omega)k_{\max}}{\sqrt{k_{\max}^2 + (\omega + \delta\omega)^2}}. \tag{2.10}$$

Otherwise, the solution of the min-max problem (2.8) is

$$\overline{k} = \frac{\sqrt{3\omega^4 + \delta\omega^4 + \sqrt{\omega^8 - 2\omega^4\delta\omega^4 + \delta\omega^8 + 16\omega^6\delta\omega^2}}}{\sqrt{2}\omega}. \tag{2.11}$$

Proof From (2.6) we find for $\overline{k} > \sqrt{2}\omega$ one maximum at $k_1 = \frac{\overline{k}\sqrt{\omega(\overline{k}^2 - 2\omega^2)(\sqrt{\overline{k}^2 - \omega^2} - \omega)}}{\overline{k}^2 - 2\omega^2}$. For $\omega + \delta\omega < \overline{k} < \sqrt{2}\omega$ there is one maximum at $-k_1$ and one at $k_2 = \frac{\overline{k}\sqrt{\omega(2\omega^2 - \overline{k}^2)(\sqrt{\overline{k}^2 - \omega^2} + \omega)}}{2\omega^2 - \overline{k}^2}$. But an asymptotic expansion as $k \longrightarrow \infty$ shows that in that case $R = 1 + 2\frac{2\omega^2 - \overline{k}^2}{k^2} + O(\frac{1}{k^4})$ and hence at the second maximum k_2 we have $R \geq 1$. Similarly at the first maximum we find $R \geq 1$ and hence the maxima must be outside of the numerical frequency range K to obtain a convergence rate less than one for $k \in K$. Therefore the maximum can only be attained on the boundaries of K, at $k = k_- := \omega - \delta\omega$, $k = k_+ = \omega + \delta\omega$ or $k = k_{\max}$ (not at $k = 0$, because there $R = 0$). We therefore analyze the dependence of R on $\overline{k}$ at these points. Taking a derivative, we find

$$\frac{\partial R}{\partial \overline{k}} = \frac{4k^6(\overline{k} - k)(\overline{k} + k)\overline{k}(k - \omega)(k + \omega)}{(k^4 + \omega^2\overline{k}^2 - 2\omega^2 k^2)^3}.$$

For $\overline{k} > \omega$ the denominator is positive, and for $\omega < k < \overline{k}$ the numerator is also positive, otherwise the numerator is negative. Hence at $k = k_-$ and $k = k_{\max}$ the convergence rate R decreases with increasing $\overline{k}$, whereas at $k = k_+$ the convergence rate R increases with increasing $\overline{k}$. A direct computation shows that for

$$\omega + \delta\omega \leq \overline{k} < \frac{\sqrt{3\omega^4 + \delta\omega^4 + \sqrt{\omega^8 - 2\omega^4\delta\omega^4 + \delta\omega^8 + 16\omega^6\delta\omega^2}}}{\sqrt{2}\omega}$$

we have $R(k_-, \overline{k}, \omega) > R(k_+, \overline{k}, \omega)$. Thus the maximum is at k_- or at $k_{\max}$ and can be reduced by increasing $\overline{k}$. When the convergence rate at k_- and k_+ are balanced and (2.11) holds, we are at the optimum, provided the convergence rate at $k_{\max}$ is smaller, or by a direct computation (2.9) is not satisfied. If however (2.9) holds, then the maximum is still at $k_{\max}$ and can be further decreased by increasing $\overline{k}$, while R at k_+ increases and R at k_- decreases further and becomes irrelevant. Hence in that case the optimum is attained when R at k_+ and $k_{\max}$ are balanced, which leads to (2.10). ∎

Corollary 4 *The asymptotic convergence rate for $k_{\max} = \pi/h$ as h goes to zero using the optimized parameter $\overline{k}$ given by (2.9), (2.10) and (2.10) is*

$$R(k_{\max}, \overline{k}, \omega) = 1 - 4\frac{\pi + 2\omega}{\pi}h^2 + O(h^4). \tag{2.12}$$

Proof For $k_{\max}$ large we are in the first case of Theorem 3. An expansion of $R(\pi/h,\overline{k},\omega)$ for h small with the optimal $\overline{k}$ given in (2.10) gives the result of the corollary. ■

We see that the optimized parameter $\overline{k}$ given by (2.10) leads to a superior asymptotic performance of the method than the fixed parameter $\overline{k} = \sqrt{2}\omega$ from Theorem 1. Nevertheless the result is a bit disappointing, since the asymptotic convergence rate is not better than the one of an unpreconditioned diffusion problem. In the next section we show how this rate can be substantially improved.

3 The Semi-Discrete Analytic Factorization

To relate the analytic factorization to the exact block LU decomposition of the discrete matrix operator, the x direction of $(-\omega^2-\Delta)$ was discretized in [5] and the analytic factorization (2.2) was constructed for the semi-discrete operator $(-\omega^2-\Delta_h)$ where $\Delta_h = D_x^- D_x^+ + \partial_{yy}$ with $D_x^+(x) := (x_{i+1}-x_i)/h$ and $D_x^-(x) := (x_i - x_{i-1})/h$ representing the discrete derivatives on a given mesh. Using a Fourier transform in y of $-\omega^2-\Delta_h$ as in the continuous case, the semi discrete analytic factorization was found in [5] to be

$$\mathcal{F}_y(-\omega^2-\Delta_h) = -\left(D_x^- + (\tau h - \frac{1}{h})\right)\frac{1}{h^2\tau}\left(D_x^+ - (\tau h - \frac{1}{h})\right) \tag{3.1}$$

with τ given by

$$\tau = \frac{1}{h^2} + \frac{-\omega^2+k^2}{2} + \frac{1}{2h}\sqrt{(-\omega^2+k^2)^2h^2 + 4(-\omega^2+k^2)}. \tag{3.2}$$

Note that as we take the limit for $h \longrightarrow 0$ in (3.1) we recover again the continuous analytic factorization (2.2) since the middle term disappears in the limit.

As in the continuous analytic factorization, we replace the nonlocal operator τ in (3.1) by a local approximation of the form

$$\tau_{app} = \frac{1}{h^2} + \frac{-\omega^2+k^2}{2} + \frac{1}{2h}(p+qk^2), \quad p,q\in\mathbb{C},\ \Re(q)>0, \tag{3.3}$$

which leads to two parabolic problems in the factorization. We insert the approximation τ_{app} into the factorization (3.1) and obtain the operator resulting from the approximate factorization of $-\omega^2+k^2-D_x^+D_x^-$ in the form

$$\mathcal{L}_{app} = \mathcal{F}_y^{-1}(-D^-D^+ + \tau_{app} + \frac{1}{\tau_{app}h^4} - \frac{2}{h^2}). \tag{3.4}$$

The complex numbers p and q are to be chosen so that $\mathcal{L}_{app}^{-1}\mathcal{L}$ is as close as possible to the identity except for a few frequencies which will be taken into account by the Krylov method. We find after some calculation the convergence rate in Fourier to be

$$\rho = 1 - \frac{2(-\omega^2+k^2)(2-\omega^2h^2+ph+h(h+q)k^2)}{(p-\omega^2h+(q+h)k^2)^2}, \tag{3.5}$$

where we have estimated again the discrete Fourier parameter in the x direction by 0. As in the continuous analytic factorization, we choose p such that ρ vanishes at $k=0$ and q such that ρ vanishes for some $\overline{k}>\omega$. We find after some computations

$$p = i\omega\sqrt{4-\omega^2h^2}, \quad q = \frac{2\sqrt{-4\omega^2+\omega^4h^2+4\overline{k}^2+\overline{k}^4h^2-2\overline{k}^2h^2\omega^2} - 2i\sqrt{4-\omega^2h^2}}{2\overline{k}^2}. \tag{3.6}$$

With these values we find for the modulus of the convergence rate $R := |\rho|^2$ for $h<2/\omega$ to be

$$R(k,\overline{k},\omega,h) = \frac{4k^4(k-\overline{k})^2(k+\overline{k})^2}{\left(k^2(\omega^2-k^2)\left((\omega^2-\overline{k}^2)h^2-\sqrt{(\omega^2-\overline{k}^2)(\omega^2h^2-4-h^2\overline{k}^2)}h\right)+2k^4+2\omega^2\overline{k}^2-4\omega^2k^2\right)^2}, \tag{3.7}$$

which agrees as h goes to zero with the convergence rate found for the continuous analytic factorization (2.5).

Theorem 5 *For $h < 1/\omega$ the convergence rate $R(k, \overline{k}, \omega, h)$ is bounded by one if and only if*

$$\overline{k} = \omega\sqrt{\frac{2-\omega h}{1-\omega h}}. \tag{3.8}$$

Proof As in the continuous case, at $k = \omega$ the convergence rate $R = 1$, independently of what we choose for the optimization parameter $\overline{k}$. A longer calculation shows that the derivative of R with respect to k at at $k = \omega$ is non zero, except if (3.8) holds. Hence if (3.8) does not hold, then the slope at $k = \omega$ is non-zero, and since at $k = \omega$ we have $R = 1$, $R > 1$ for some values of k and we have shown the only if part. For the if part, we assume that $\overline{k}$ is given by (3.8). Again using the derivative of R with respect to k we find two minima at $k = 0$ and $k = \overline{k}$ and one maximum at $k = \omega$ where $R = 1$ as always at $k = \omega$. It remains to analyze R as $k \longrightarrow \infty$. Taking the limit, we find

$$\lim_{k\to\infty} R(k, \overline{k}, w, h) = (1-\omega h)^2 < 1$$

under the condition on h, which concludes the proof. ■

Corollary 6 *The asymptotic convergence rate $R(k_{max}, \overline{k}, \omega, h)$ with $k_{\max} = \pi/h$ and optimization parameter $\overline{k}$ given by (3.8) is for small h given by*

$$R(k_{max}, \overline{k}, \omega, h) = 1 - 2\omega h + O(h^2). \tag{3.9}$$

Proof The result follows by expanding $R(\pi/h, \overline{k}, \omega, h)$ with $\overline{k}$ given by (3.8) for h small. ■
Note that this result is a big improvement over the results for the continuous factorization. As in the continuous case however we can try to optimize the convergence rate solving the min-max problem (2.8) for the convergence rate R given in (3.7). If we choose the same strategy as in the continuous case for h small, we get the following theorem.

Theorem 7 *With optimized parameter $\overline{k}$ defined by*

$$R(\omega + \delta\omega, \overline{k}, \omega, h) - R(k_{\max}, \overline{k}, \omega, h) = 0 \tag{3.10}$$

the asymptotic convergence rate for $k_{\max} = \pi/h$ as h goes to zero is

$$R(k_{\max}, \overline{k}, \omega, h) = 1 - 2\sqrt{\omega^2 + 4\omega\delta\omega + 2\delta\omega^2}h + O(h^2). \tag{3.11}$$

Proof There is no closed form solution for $\overline{k}$ satisfying (3.10), but we know from (2.10) of the continuous analytic factorization that as $h \longrightarrow 0$ the optimal parameter is $\overline{k} = \sqrt{2}(\omega + \delta\omega)$. We therefore insert the ansatz $\overline{k} = \sqrt{2}(\omega + \delta\omega) + Ch^\alpha$, $\alpha > 0$ and $k_{\max} = \pi/h$ into (3.10) and expand for small h. We find for (3.10) after a long calculation asymptotically

$$\frac{2\omega^2\sqrt{\omega^2 + 4\omega\delta\omega + 2\delta\omega^2}}{(\omega+\delta\omega)^2}h + 4\frac{\sqrt{2}(2\omega+\delta\omega)\delta\omega C}{(\omega+\delta\omega)^3}h^\alpha + O(h^2 + h^{1+\alpha} + h^{2\alpha}) = 0.$$

Balancing the first terms leads to $\alpha = 1$ and

$$C = -\frac{\sqrt{\omega^2 + 4\omega\delta\omega + 2\delta\omega^2}\omega^2(\omega+\delta\omega)}{2\sqrt{2}\delta\omega(2\omega+\delta\omega)}.$$

Inserting this asymptotic result for $\overline{k}$ into R and evaluating at $k = k_{\max}$ gives the result of the theorem. ■
Figure 1 shows a comparison of the convergence rates, for $\omega = 10\pi$ and $\delta\omega = \pi$. On the left R is shown as a function of k for $h = 1/50$ for the continuous analytic factorization with fixed $\overline{k}$ in red and with optimized $\overline{k}$ in green and for the semi-discrete analytic factorization with fixed $\overline{k}$ in yellow and optimized $\overline{k}$ in blue. One can clearly see the superior performance of the semi-discrete factorization given by the lower curves and also a small difference between the performance of the two continuous factorizations in red and green. On the right with the same color coding we show for the same convergence rates $1 - R$ at $k = k_{\max}$ as a function of h. The asymptotic behavior is as predicted by the analysis: the semi-discrete factorization has the weakest dependence on h, $R = 1 - O(h)$ and using the fixed or optimized $\overline{k}$ does not make a significant difference (yellow and blue), whereas for the continuous factorization the difference is significant, $1 - O(h^4)$ in red and $1 - O(h^2)$ in green.

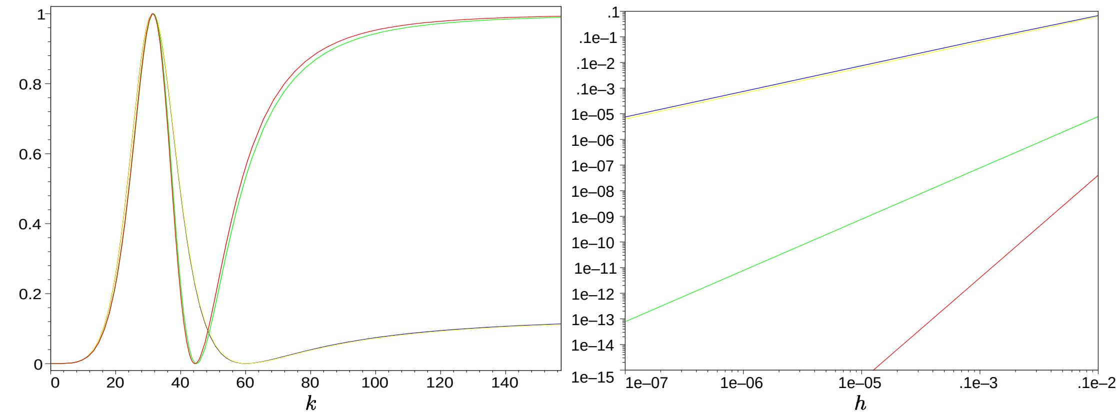

Figure 1: On the left comparison of the convergence rates as a function of k and on the right a comparison of $1 - R$ as a function of h where R attains its maximum.

4 Conclusions

We have analyzed two versions of the AILU preconditioner for the Helmholtz equation, a continuous one and a semi-discrete one. We have derived the optimal parameters for the performance of the preconditioners and we have shown that the semi-discrete AILU has a much better performance than the continuous AILU for acoustic problems.

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