“ANALYSIS OF SAW DIFFRACTION BY PERIODIC MEDIA WITH MULTI-WAVE COUPLED-WAVE THEORY”

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ABSTRACT
The grating structure as the most fundamental element of SAW (Surface Acoustic Wave) devices is discussed. The modeling based on the periodic modulation of basic properties is included. Diffraction characteristics of general planar (slab) gratings and surface-relief (corrugated) gratings are presented. An exact formulation of grating diffraction problem without approximations (rigorous coupled-wave theory) is proposed. Starting from the wave equation, rigorous form of analysis with coupled-wave theory is developed.

I. INTRODUCTION

Diffraction of SAW (surface acoustic wave) by periodic structures is of increasing importance in an expanding variety of engineering applications. Grating diffraction is central in the fields of acoustooptics, integrated optics, holography, data processing, and spectral analysis.
Gratings may be planar (slab) gratings. The periodic modulation may be in the permittivity (or equivalently index of refraction) or in the conductivity (or equivalently absorption) or combination of these. Also gratings may be of the surface-relief (corrugated) type with periodic variations in the surface of material.

Diffraction of SAW by spatially periodic media may be analysed by numerous methods and with a wide variety of possible assumptions. The method of multi-wave coupled-wave theory of analysing grating diffraction is used. The coupled-wave approach is confusingly also sometimes called coupled-mode approach. Starting from the wave equation, rigorous form of analysis with coupled-wave theory will be developed.

II. ANALYSIS OF FIELD INSIDE GRATING

A. Field Representation

A plane grating in the region from z = 0 to z = d, as depicted in Fig.1, can be described by

$$\rho(x') = \rho_0 + \rho_1 \cos Kx' \tag{1}$$

where ρ_0 is the average mass density in the grating region, ρ_1 is the amplitude of the sinusoidal relative density, ϕ is grating slant angle, and $K = 2\pi/\Lambda$, where Λ is the grating period. For an incident plane wave, the wave equation is

$$\nabla^2 A(x', z') + k^2 \rho(x') A(x', z') = 0 \tag{2}$$

where $k = 2\pi/\lambda$ and $A(x', z')$ is the total field inside grating. The field in the grating region may be expressed in terms of "modes", thus the total field may be written as

$$A(x', z') = \sum_{\nu=-\infty}^{\infty} A_\nu (x', z') \tag{3}$$

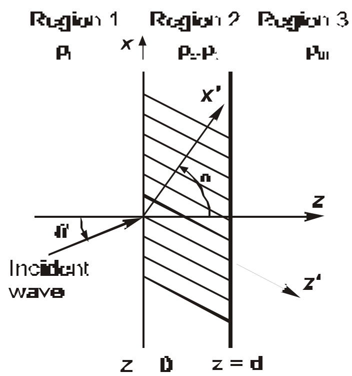

Fig. Planar slanted-fringe grating geometry

The field corresponding to a particular mode n, may be assumed to be expressible as a product so that

$$A_\nu (x', z') = U_\nu X_\nu (x') Z_\nu (z'). \tag{4}$$

Upon substitution of this assumed solution into (2), and dividing by $A_n(x',z')$, separation of variables in the wave equation is achieved. Thus the x' and the z' part must be equal to a constant. Letting the constant to be $-\xi^2_\nu$, the solution for $Z_\nu(z')$ may be written as

$$Z_\nu (z') = B_\nu \exp(-j\xi_\nu z'). \tag{5}$$

The x' equation resulting from separation of variables is

$$\frac{d^2 X_\nu(x')}{dx'^2} + \left[k^2 \rho(x') - \xi_\nu^2\right] X_\nu (x') = 0 \tag{6}$$

or

$$\frac{d^2 X_\nu(x')}{dx'^2} + (a_1 + a_2 \cos Kx') X_\nu (x') = 0 \tag{7}$$

where $a_1 = k^2 \rho_0 - \xi_\nu^2$ and $a_2 = k^2 \rho_1$. The general solution of this equation was found by Floquet to be

$$X_\nu (x') = \Phi_\nu (x') \exp(-j\beta_\nu x') \tag{8}$$

where β_ν is a phase factor and $\Phi_\nu (x')$ is periodic in x' with period Λ. That is,

$$\Phi_\nu (x') = \Phi_\nu (x' + \Lambda) \tag{9}$$

for any x'. Since $\Phi_\nu (x')$ is periodic, it may be expanded in a Fourier series as

$$\Phi_\nu(x') = \sum_{i=-\infty}^{+\infty} C_{\nu i} \exp(jiKx') \tag{10}$$

and so $X_\nu(x')$ may be written as

$$X_\nu(x') = \sum_{i=-\infty}^{+\infty} C_{\nu i} \exp(-j\beta_{\nu i} x') \tag{11}$$

where

$$\beta_{\nu i} = \beta_\nu - iK\ . \tag{12}$$

Equation (12) is often referred as the "Floquet condition". Each modal field, $A_\nu(x',z') = U_\nu X_\nu(x') Z_\nu(z')$, may thus be expressed as

$$A_\nu(x',z') = D_\vartheta \exp(-j\xi_\nu z') \sum_{i=-\infty}^{+\infty} C_{\nu i} \exp(-j\beta_{\nu i} x') \tag{13}$$

where $D_\nu = U_\nu B_\nu$. Rotating from coordinate system of the grating (x',z') to the coordinate system of the boundary (x, z), the total field may be expressed as

$$A_\nu(x,z) = D_\nu \sum_{i=-\infty}^{+\infty} C_{\nu i} \exp\{-j[(\beta_\nu - iK)\sin\phi - \xi_\nu \cos\phi]x\} \cdot \exp\{-j[\xi_\nu \sin\phi + (\beta_\nu - iK)\cos\phi]z\}. \tag{14}$$

The normalized field of the incident plane wave is given by

$$A_i = \exp(-j\bar{k}_1 \cdot \bar{r}) = \exp[-j(k_{1x}x + k_{1z}z)] \tag{15}$$

where $k_{1x} = k\rho_1^{1/2} \sin\theta'$, and $k_{1z} = k\rho_1^{1/2}\cos\theta'$. For the limit case of zero grating modulation (ρ_1 ®0), the i = 0 undiffracted field of each mode is phased matched to the incident field at the z = 0 boundary. That is

$$k_{1x} = k_{2x} \tag{16}$$

where $k_{2x} = k\rho_o^{1/2} \sin\theta$, and θ is the angle of refraction of the incident wave in the second region. From (14), the phase-matching condition is thus

$$k\rho_1^{1/2} \sin\theta' = \beta_\nu \sin\phi - \xi_\nu \cos\phi\ . \tag{17}$$

The total field inside grating is represented by the sum of all of the individual modal fields as

$$A(x,y) = \sum_{\nu=-\infty}^{+\infty} A_\nu(x,z) \tag{18}$$

and so the total field is

$$A(x,z) = \sum_{\nu=-\infty}^{+\infty} D_\nu \sum_{i=-\infty}^{+\infty} C_{\nu i} \exp[-j(k_{2x} - iK\sin\phi)x] \cdot \exp\{-j[\xi_\nu \sin\phi + (\beta_\nu - iK)\cos\phi]z\}. \tag{19}$$

This is a general form for the field inside grating, produced by an incident plane wave. This equation is rigorously derived (without approximations). The total field can be rewritten as a sum over the space-harmonic components i, and coupled-wave field expansion can be obtained from it.

B. Coupled-Wave Expansion and Resulting Equations

Interchanging the order of the summations in (19), the total field inside grating may be rewritten

$$A(x,z) = \sum_{i=-\infty}^{+\infty} \exp[-j(k_{2x} - iK\sin\phi)x] \cdot \sum_{\nu=-\infty}^{+\infty} D_\nu C_{\nu i} \exp\{-j[\xi_\nu \sin\phi + (\beta_\nu - iK)\cos\phi]z\}. \tag{20}$$

Performing the summation over the modes ν, the quantity $\hat{S}_i(z)$ may be defined as

$$\hat{S}_i(z) \equiv \sum_{\nu=-\infty}^{+\infty} D_\nu C_{\nu i} \exp\{-j[\xi_\nu \sin\phi + (\beta_\nu - iK)\cos\phi]z\} \qquad (21)$$

and this is a function of z only. A new function of $S_i(z)$ may be defined as

$$S_i(z) \hat{=} \hat{S}_i(z)\exp[+j(k_{2z} - iK\cos\phi)z] \qquad (22)$$

so that the field may be expressed as

$$A(x,z) = \sum_{i=-\infty}^{+\infty} S_i(z)\exp\{-j[k_{2x} - iK\sin\phi)x + (k_{2z} - iK\cos\phi)z]\} \qquad (23)$$

or in vector notation

$$A(x,z) = \sum_{i=-\infty}^{+\infty} S_i(z)\exp[-j(\bar{k}_2 - i\bar{K})\cdot\bar{r}] \qquad (24)$$

where $\bar{k}_2$ would be the wavevector of the refracted incident wave in the absence of grating modulation. This form of the total field is more useful, since leads to constant-coefficient coupled-wave differential equations for general slanted gratings. This form of the expansion express the total field as the sum of inhomogeneous plane waves having wavevectors given by the vector Floquet condition

$$\bar{\sigma}_i = \bar{k}_2 - i\bar{K}\ . \qquad (25)$$

The expansion (24) has great intuitive appeal. The incident homogeneous plane wave may be visualized as being divided into many diffracted inhomogeneous plane waves that have directions given bay (25), the vector Floquet condition for unbounded periodic medium. The phase fronts of inhomogeneous plane waves i = 1 to i= +2 are depicted in Fig.2 together with the corresponding Floquet condition (shown in inset).

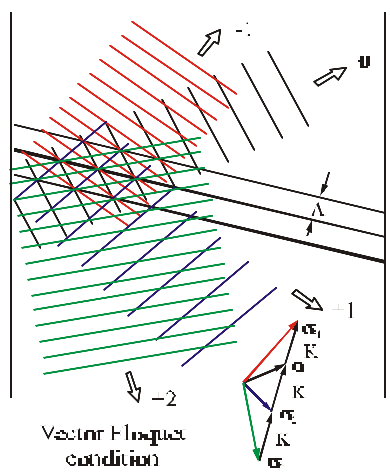

Fig. 2. Visualization of inhomogeneous plane waves inside the grating

The i = 0 inhomogeneous plane wave corresponds to the refracted incident wave. In this expansion, the diffracted waves form an inference pattern with the incident wave that has a periodicity Λ and slant angle ϕ that are the same as grating. Substituting the expansions (23) and (1) into the wave equation, and performing the indicated differentiations gives

$$\frac{1}{2\pi^2}\frac{d^2\hat{S}_i(z)}{dz^2} - j\frac{2}{\pi}\left[\frac{(\rho_0)^{1/2}\cos\theta}{\lambda} - \frac{i\cos\phi}{\Lambda}\right]\frac{d\hat{S}_i(z)}{dz} + \frac{2i(m-i)}{\Lambda^2}\hat{S}_i(z) + \frac{\rho_1}{\lambda^2}\left[\hat{S}_{i-1}(z) + \hat{S}_{i+1}(z)\right] = 0 \qquad (26)$$

where the quantity m has been defined as

$$m = \frac{2\Lambda(\rho_0)^{1/2}}{\lambda}\cos(\theta - \phi) \qquad (27)$$

These rigorous coupled-wave equations represent a set of second-order linear differential equations with constant coefficients. Using the methods of linear system analysis this equations system may be transformed into a state-space description. By defining the state variables as

$$S_{1,i}(z) = S_i(z)$$
$$S_{2,i}(z) = \frac{dS_i(z)}{dz} \tag{28}$$

the infinite set of second order differential equations (26) are transformed into infinite sets of first-order differential equations

$$\frac{dS_{1,i}(z)}{dz} = S_{2,i}(z)$$
$$\frac{dS_{2,i}(z)}{dz} = -\frac{2\pi^2\rho_1}{\lambda^2}S_{1,i-1}(z) + \frac{4\pi^2(i-m)}{\Lambda^2}S_{1,i}(z) - \frac{2\pi^2\rho_1}{\lambda^2}S_{1,i+1}(z) + j4\pi\left[\frac{\rho_0^{1/2}\cos\theta}{\lambda} - \frac{i\cos\phi}{\Lambda}\right]S_{2,i}(z) \tag{29}$$

Equations (29) are the state equations corresponding to the rigorous coupled-wave equations (26). These constituent sate equations may be written in matrix form as

$$\dot{\boldsymbol{S}} = \boldsymbol{AS} \tag{30}$$

where $\dot{\boldsymbol{S}}$ and $\boldsymbol{S}$ are the column vectors and $\boldsymbol{A}$ is the coefficient matrix.

III. SURFACE-RELIEF GRATINGS

A. <u>Problem Formulation</u>

Surface-relief (corrugated) gratings can be rigorously analysed using coupled-wave analysis. This can be done by dividing the surface-relief grating into large number of the thin (planar) layers. Each layer is then analysed using the state variables method of solution of the rigorous coupled-wave equations for that grating. There are no approximations in the analysis and results are obtainable to any arbitrary level of accuracy.

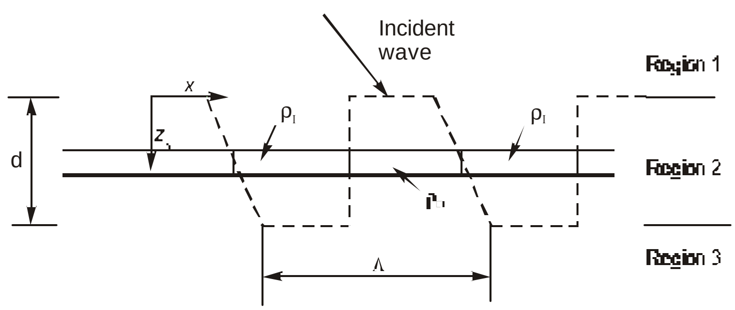

Fig.3 The *n*th planar grating resulting from decomposition of a surface-relief grating.

Region 1 and 3, as shown in Fig. 3, are homogeneous with densities ρ_{I} and ρ_{III} , respectively. The region 2 (the grating region) consists of a periodic distribution of both types of materials. The boundary between region 1 and 3 in region 2 is given by

$$z = F(x)F(x+\Lambda) \tag{31}$$

where Λ is the grating period. The function $F(x)$ represents the grating surface profile, and there are no restrictions on the form of $F(x)$ in this analysis.

B. <u>Coupled-Wave Expansion and Resulting Equations</u>

In the present analysis, the grating region is divided into N thin planar gratings slabs perpendicular to the z axis. Then the rigorous coupled-wave analysis that has been developed for planar gratings is applied to each slab grating. If the individual planar grating are sufficiently thin, any grating profile can be analysed. The relative density for nth slab grating is periodic, $\rho_n(x,z_n)=\rho_n(x+\Lambda,z_n)$, and may be expanded in a Fourier series as

$$\rho_n(x,z_n)=\rho_{\mathrm{I}}+(\rho_{\mathrm{III}}-\rho_{\mathrm{I}})\sum_{h=-\infty}^{\infty}\tilde{\rho}_{h,n}e^{jhKx} \tag{32}$$

where z_n is the z coordinate of the nth slab, h is the harmonic index, K is the magnitude of the grating vector ($K=2\pi/\Lambda$), and $\tilde{\rho}_{h,n}$ are the normalized complex harmonic amplitude coefficients given by

$$\tilde{\rho}_{h,n}=\frac{1}{\Lambda}\int_0^{\Lambda}dxf(x,z_n)\exp(-jhKx)dx \tag{33}$$

where $f(x,z_n)$ is equal to either zero or unity depending whether, for particular value of x, the grating relative density is ρ_{I} or ρ_{III}, respectively. Using equation (24) the field in nth slab may be expressed as

$$A_{2,n}(x,z)=\sum_{i=-\infty}^{\infty}S_{i,n}(z)\exp[(\bar{k}_{2,n}-i\bar{K})\cdot\bar{r}] \tag{34}$$

where $\bar{K}=K\hat{x}$ and $\bar{k}_{2,n}=2\pi(\rho_{0,n})^{1/2}/\lambda$ is the wave-vector of the zero-order refracted wave, and $\rho_{0,n}$ is the average relative density for nth slab grating. Substituting (33) and (34) into wave equation, performing the indicated differentiations, and setting the coefficient of each exponential term equal to zero for nontrivial solutions yields the coupled-wave equations for nth slab grating

$$\frac{d^2S_{i,n}(z)}{dz^2}-j2\left(k_{2,n}^2-k_1^2\sin^2\theta'\right)^{1/2}\frac{dS_{i,n}(z)}{dz}+K^2i(m-i)S_{i,n}(z)$$
$$+k^2(\rho_{III}-\rho_I)\sum_{h=1}^{\infty}\left[\tilde{\rho}_{h,n}S_{i-h,n}(z)+\tilde{\rho}_{h,n}^{*}S_{i+h,n}(z)\right]=0 \tag{35}$$

These coupled-wave equations are analogous to (26).

IV. SUMMARY

A rigorous analysis of diffraction by slab gratings and surface-relief gratings has been presented. The field inside the modulated medium is expanded in terms of the space-harmonic components (i) of the field in periodic structure. By selecting appropriate coupled-wave expansion, the amplitudes of space-harmonic components of the field have been shown to be directly obtainable using a state-variables approach from linear system theory.

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