DETERMINATION OF THE ENERGY VELOCITY OF A LAMB WAVE BY MEANS OF FREQUENCY AND ANGULAR QUALITY FACTORS

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ABSTRACT

The product frequency-frequency phase derivative of the reflection coefficient of an immersed elastic plate, calculated at a resonance frequency, is proportional to the frequency Q_x factor. Denoting as y the incident angle sine, the product y-phase derivative with respect to y corresponds to an angular Q_y factor. From the expressions of the different mean energies related to wave propagation and the definition of the energy velocity depending on the ratio of the average power flow through the plate thickness to the average total stored energy, we can express the energy velocity as a function of Q_x, Q_y and the phase velocity.

INTRODUCTION

The subject of this paper is based upon the combination of two methods of investigation of guided waves in an elastic plate immersed in a fluid. From an energy point of view, we classically obtain the guided wave energy velocity by calculating the ratio of the average power flow through the plate thickness to the average stored energy per unit length of the plate, according to Auld [1]. This energy velocity is said to be equal to the group velocity; it is, in most cases, numerically true. We can also study the resonant aspect associated to the guided wave propagation, which involves radiated energy into the fluid. The Phase Gradient Method (PGM) consists in studying the partial derivatives of the phase of the reflection coefficient of the plate, with respect to the frequency, the incident angle and the phase velocities of the different waves involved in the problem. This method is proved to be efficient to characterize the resonances of the plate [2]. It is shown in this paper, that, on the one hand, the PGM leads to the introduction of frequency, angular, longitudinal and transverse quality factors, and, on the other hand, the group velocity can be recovered using the frequency and angular quality factors. Using their energy definitions, we can relate these quality factors to different average stored energies. In addition, we show that the energy velocity can also be obtained thanks to the frequency and angular energy quality factors. In a first part, we briefly recall the basis of the PGM and define the different quality factors. In a second part, the expression of the group velocity as a function of the quality factors is demonstrated. In a third part, their energy expressions are established. In a fourth part, the relation linking the energy velocity and the quality factors is presented.

I BASIS OF THE PHASE GRADIENT METHOD (PGM)

Let us consider an elastic plate, immersed in a fluid, insonified at the incidence angle θ, by an harmonic plane wave of frequency f. The plate thickness is d, its density ρ_S, the phase velocities of the longitudinal and transverse waves propagating in the plate are $c_{L,T}$. In the fluid, the density is ρ_F, the phase velocity is c_F. For the numerical results, we consider a 5 mm-thick aluminum plate immersed in water. The parameter values are: $\rho_S = 2800\ \text{kg/m}^3$, $c_L = 6380\ \text{m/s}$, $c_T = 3100\ \text{m/s}$, $\rho_F = 1000\ \text{kg/m}^3$ and $c_F = 1470\ \text{m/s}$.

The factorized expression of the reflection coefficient of the immersed plate can be written as:

$$\underline{R} = \frac{C_A C_S - \tau^2}{\left(C_A + j\tau\right)\left(C_S - j\tau\right)}.$$

The nullity of the functions $C_{A,S}$ gives the antisymmetric and symmetric vibration modes of the plate in vacuum and τ is the ratio of the acoustic impedances in the fluid and in the plate.

The phase ϕ of the reflection coefficient can be written as:

$$\phi = \operatorname{atan}\left(\frac{\tau}{C_S}\right) - \operatorname{atan}\left(\frac{\tau}{C_A}\right).$$

The Phase Gradient Method consists in studying the partial derivatives of the phase with respect to the frequency f, the sine of the incidence angle, y, and the phase velocities $c_{L,T,F}$. This method is based on a fundamental relation [2, 3], linking the different partial derivatives:

$$f\frac{\partial\phi}{\partial f} + c_L\frac{\partial\phi}{\partial c_L} + c_T\frac{\partial\phi}{\partial c_T} + c_F\frac{\partial\phi}{\partial c_F} = 0.$$

It is valid, whatever the frequency f and the incident angle θ. Moreover, we can show, in the vicinity of a resonance frequency, f_{Res}, the following relation:

$$c_F\frac{\partial\phi}{\partial c_F} = -y\frac{\partial\phi}{\partial y}.$$

The resonance frequency f_{Res} corresponds to the real part of a frequency pole of $\underline{R}$ and are denoted as

$$\underline{f}_P = f_{Res} - j\frac{\Gamma}{2}$$

where Γ is the resonance width.

We can show that an approximated expression of the function $f\frac{\partial\phi}{\partial f}(f)$ can be written as:

$$\left(f\frac{\partial\phi}{\partial f}\right)_{app}(f) = \frac{f_{Res}\frac{\Gamma}{2}}{\left(f - f_{Res}\right)^2 + \left(\frac{\Gamma}{2}\right)^2}.$$

So, when $f = f_{Res}$, one has

$$\left(f\frac{\partial\phi}{\partial f}\right)_{app}\left(f_{Res}\right) = 2\frac{f_{Res}}{\Gamma} = 2Q_x.$$

Q_x is the classical frequency quality factor. In Figure 1, following the A_{11} mode dispersion curve, we have plotted the evolutions of the frequency quality factor Q_x, obtained either by the frequency phase derivative (solid line) or by the frequency poles $\underline{f}_P$ (dashed line); the agreement is very good.

By analogy, all the functions of the type $u\frac{\partial\phi}{\partial u}$, $u = c_{L,T,F}$, y, calculated at f_{Res}, are also proportional to quality factors. In particular, we can write

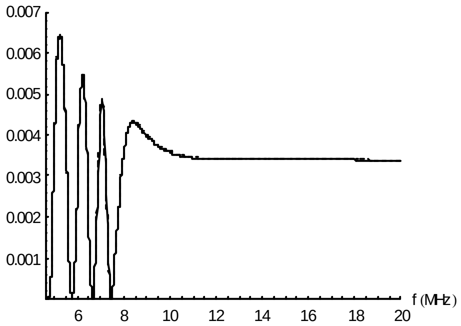
$$-y\frac{\partial\phi}{\partial y}\left(f_{Res}\right) = 2Q_y.$$

the angular phase derivative being negative, in most cases.

Q_y is qualified as an angular quality factor. It corresponds to the ratio of the real part to the imaginary part of the angular pole of $\underline{R}$, calculated at f_{Res}. Denoting an angular pole as $\underline{y}_P = y_P + j\frac{\gamma_P}{2}$, the angular quality factor can be written as:

$$Q_y = \frac{y_P}{\gamma_P}.$$

In Figure 2, still for the A_{11} mode, we have plotted the evolutions of the angular quality factor Q_y, obtained either by the angular phase derivative (solid line), or by the angular poles $\underline{y}_P$ (dashed line); the comparison is still very good.



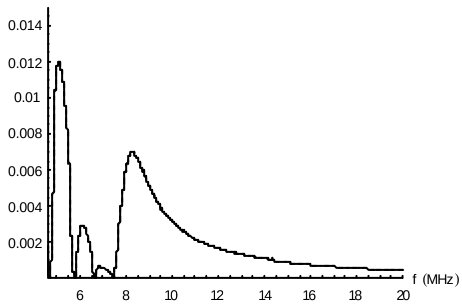


Figure 1

Evolutions of $1/Q_x$ for the A_{11} mode, obtained by the frequency phase derivative (solid line) and by the frequency poles (dashed line)

Figure 2:

Evolutions of $1/Q_y$ for the A_{11} mode, obtained by the angular phase derivative (solid line) and by the angular poles (dashed line)

II EVALUATION OF THE GROUP VELOCITY WITH THE QUALITY FACTORS Q_X AND Q_Y

The group velocity, v_g, is classically defined as

$$v_g = \frac{\partial\omega}{\partial K_x}.$$

where ω is the angular frequency and K_x is the wave number of the guided wave propagating along the plate, at a resonance frequency f_{Res}.
K_X can be written as

$$K_x = \frac{2\pi f_{Res} y}{c_F}.$$

where, according to the Cremer's coincidence rule, the ratio $\frac{c_F}{y}$ corresponds to the phase velocity v_x.
So, the group velocity can be written as

$$v_g = \frac{c_F}{y + f_{Res}\left(\frac{\partial y}{\partial f}\right)_{f_{Res}}}.$$

It was shown [2], that

$$\left(\frac{\partial y}{\partial f}\right)_{f_{Res}} = \frac{\gamma_P}{\Gamma}.$$

In Figure 3, following the dispersion curve of mode A_{11}, we have plotted the evolutions of $\left(\frac{\partial y}{\partial f}\right)_{f_{Res}}$ (solid line) and $\frac{\gamma_P}{\Gamma}$ (dashed line): the validity of the previous relation is numerically proved.
Using the definitions of Q_x and Q_y, we may now write the following approximated relation:

$$v_g = \frac{Q_y}{Q_x + Q_y} v_x .$$

In Figure 4, we have plotted the evolutions of the group velocity, obtained either from its classical definition (solid line) or from the previous relation (dashed line). The agreement is good.

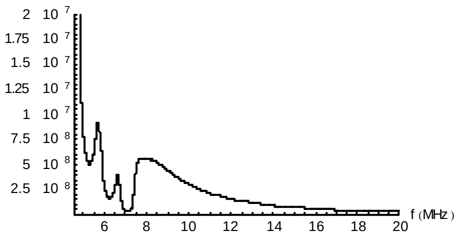


Figure 3

Evolutions of $\left(\frac{\partial y}{\partial f}\right)_{f_{Res}}$ (solid line)

and $\frac{\gamma_P}{\Gamma}$ (dashed line) (A_{11} mode)

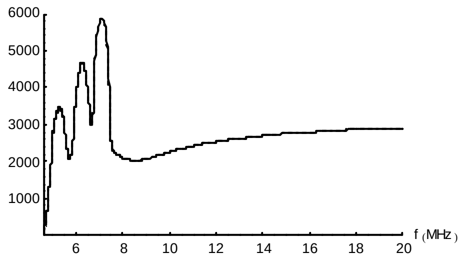


Figure 4

Plots of the exact v_g (solid line)

and the approximated one (dashed line)

III IDENTIFICATION OF Q_X AND Q_Y TO ENERGY EXPRESSIONS

The energetic definition of a quality factor is

$$Q = \frac{\omega \langle \text{stored Energy} \rangle}{\langle \text{Dissipated Power} \rangle}$$

where the brackets indicate that we deal with average values.
Therefore, we establish the expressions of the average kinetic and potential energies stored in an elementary volume dV in the plate, of length dx. Without detailing, we can show that the average kinetic and potential energies are identical, as for a plate in vacuum [4]. So, in the following, we only consider kinetic energies. The average kinetic energies associated to the longitudinal and transverse waves in the plate are obtained by

$$\left\langle E_{K_{L,T}} \right\rangle = \frac{1}{4} \rho_S dx \operatorname{Re}\left(\int_{z=-\frac{d}{2}}^{z=\frac{d}{2}} \left(\underline{\vec{v}}_{L,T} \underline{\vec{v}}^{*}_{L,T} \right) dz \right).$$

We can decompose these energies into one term corresponding to a displacement along the x-direction (parallel to the plate) and another term corresponding to a displacement along the z-direction (normal to the plate). We denote them as

$$\left\langle E_{K_{L,T_{x,z}}} \right\rangle = \frac{1}{4} \rho_S dx \operatorname{Re}\left(\int_{z=-\frac{d}{2}}^{z=\frac{d}{2}} \left(\underline{v}_{L,T_{x,z}} \underline{v}^{*}_{L,T_{x,z}} \right) dz \right).$$

We can show, that :

$$\begin{cases} \left\langle E_{K_{Lx}} \right\rangle = \left\langle E_{K_L} \right\rangle \sin^2 \theta_L \\ \left\langle E_{K_{Lz}} \right\rangle = \left\langle E_{K_L} \right\rangle \cos^2 \theta_L \end{cases}, \quad \begin{cases} \left\langle E_{K_{Tx}} \right\rangle = \left\langle E_{K_T} \right\rangle \cos^2 \theta_T \\ \left\langle E_{K_{Tz}} \right\rangle = \left\langle E_{K_T} \right\rangle \sin^2 \theta_T \end{cases}.$$

We may interpret the energies $\left\langle E_{K_{Lx}} \right\rangle$ and $\left\langle E_{K_{Tz}} \right\rangle$ as average energies respectively associated to guided longitudinal and transverse waves in the plate and the energies $\left\langle E_{K_{Lz}} \right\rangle$ and $\left\langle E_{K_{Tx}} \right\rangle$ as average energies associated to longitudinal or transverse standing waves in the plate thickness.

We also introduce average interaction energies between longitudinal and transverse waves, defined as:

$$\left\langle I_{K_{x,z}} \right\rangle = \frac{1}{4}\rho_S dx \mathrm{Re}\left(\int_{z=-\frac{d}{2}}^{z=\frac{d}{2}} \left(\underline{v}_{L_{x,z}} \underline{v}^{*}_{T_{x,z}} + \underline{v}^{*}_{L_{x,z}} \underline{v}_{T_{x,z}} \right) dz \right).$$

The dissipated power, $\left\langle P_d \right\rangle$, is obtained by calculating the z-components of the Poynting vector at the interfaces between the plate and the surrounding fluid. One has

$$\left\langle P_d \right\rangle = \rho_F dx\omega^4 \frac{\cos\theta}{c_F}\left(1-\left|\underline{R}\right|^2\right).$$

We introduce the following notations

$$\begin{cases} \left\langle E_{\parallel} \right\rangle = 2\left(\left\langle E_{K_{L_x}} \right\rangle + \left\langle E_{K_{T_z}} \right\rangle + \left\langle I_{K_x} \right\rangle\right) \\ \left\langle E_{\perp} \right\rangle = 2\left(\left\langle E_{K_{L_z}} \right\rangle + \left\langle E_{K_{T_x}} \right\rangle + \left\langle I_{K_z} \right\rangle\right) \end{cases}.$$

We can show analytically that the frequency quality factor may be obtained by

$$Q_x = \frac{\omega\left\langle E_{\perp} \right\rangle}{\left\langle P_d \right\rangle}$$

and the angular quality factor by

$$Q_y = \frac{\omega\left\langle E_{\parallel} \right\rangle}{\left\langle P_d \right\rangle}.$$

The frequency quality factor is explicitly related to standing waves across the plate and the angular quality factor to guided waves along the plate. So, $Q_x/(Q_x+Q_y)$ can be viewed as a stationnary wave ratio and $Q_y/(Q_x+Q_y)$ as a guided wave ratio. In Figures 5 and 6, we compare the evolutions of $Q_{x,y}$ obtained either by the frequency and angular phase derivatives (solid lines), or from their energy expressions (dashed lines). The comparison of the Q_x plots is very good, the one of the Q_y is also quite good, except in the vicinity of the longitudinal critical angle.

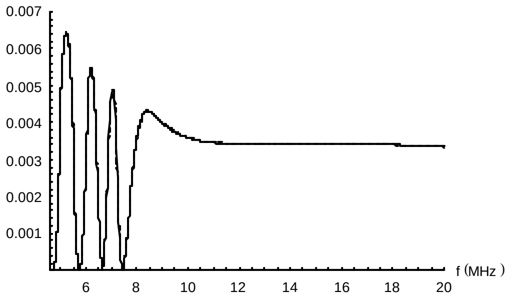


Figure 5

Evolutions of $1/Q_x$ for the A_{11} mode, obtained by the frequency phase derivative (solid line) and by its energy expression (dashed line)

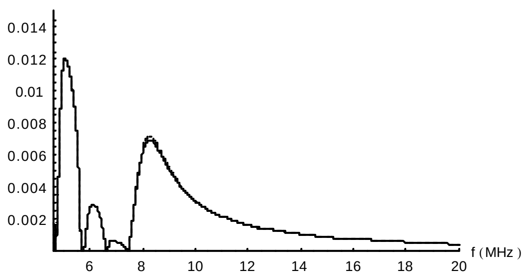


Figure 6

Evolutions of $1/Q_y$ for the A_{11} mode, obtained by the angular phase derivative (solid line) and by its energy expression (dashed line)

IV EVALUATION OF THE ENERGY VELOCITY

We classically define the energy velocity, v_e, as the ratio of the average power flow through the plate thickness to the average total stored energy. We can write

$$v_e = \frac{\left\langle P_x \right\rangle}{\left\langle E_{tot} \right\rangle} = \frac{\left\langle P_x \right\rangle}{2\left\langle E_K \right\rangle}$$

where $\left\langle P_x \right\rangle$ is the average value of the x-component of the Poynting vector calculated on a right section of the plate and $\left\langle E_K \right\rangle$ is the average total kinetic energy.

We have

$$2\langle E_K\rangle = 2\left(\langle E_{K_L}\rangle + \langle E_{K_T}\rangle + \langle I_{K_x}\rangle + \langle I_{K_z}\rangle\right) = \langle E_{\parallel}\rangle + \langle E_{\perp}\rangle .$$

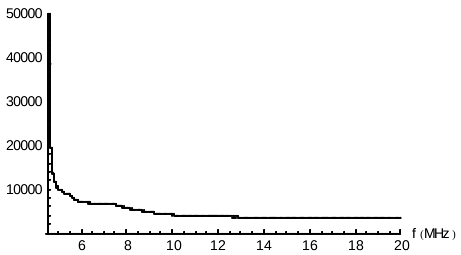
We verify numerically that the ratio $\langle P_x\rangle / \langle E_{\parallel}\rangle$ is nearly equal to the phase velocity v_X, as shown in Figure 7. So, the energy velocity can be written

$$v_e = \frac{\langle P_x\rangle}{\langle E_{\parallel}\rangle} \frac{\langle E_{\parallel}\rangle}{\left(\langle E_{\parallel}\rangle + \langle E_{\perp}\rangle\right)}$$

and using the expressions of $Q_{x,y}$, may be approximated by

$$v_e = \frac{Q_y}{Q_x + Q_y} v_x .$$

We obtain the same formula as for the group velocity.
In Figure 8, we compare the plots of the energy velocity obtained either by its classical definition (solid line), or by the previous relation (dashed line); the agreement is good.



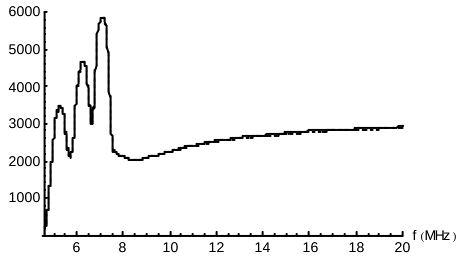


Figure 7

Plots of the phase velocity c_X (solid line) and of $\langle P_x\rangle / \langle E_{\parallel}\rangle$ (dashed line)

Figure 8

Plots of the exact energy velocity (solid line) and the approximated one (dashed line)

V CONCLUSION

In this paper, it was shown that, the group velocity, as well as the energy velocity, can be obtained by means of frequency and angular quality factors. They can be easily obtained using the Phase Gradient Method. The energy developments presented show that these velocities, numerically equal, can be written as the product of the guided wave ratio $Q_y/(Q_x+Q_y)$ and the phase velocity.

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