# Acoustic scattering by two impenetrable cylinders: asymptotic approach and physical interpretation

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ABSTRACT

Acoustic scattering by two identical, impenetrable and parallel cylinders is studied by emphasizing the role of the symmetries of the scatterer. The characteristic determinant of the scattering matrices is expanded in terms of simple traces which are semiclassically evaluated in order to extract the periodic orbits. Generalized formulas are derived for all the contributions which are purely geometrical or composite (including a creeping section). All the scattering resonances, interpreted as periodic orbits, are in excellent agreement with the exact results. The problem of scattering by two impenetrable cylinders can be considered as a canonical problem.

Scattering problems by open systems have been extensively investigated in many fields of physics (for instance in quantum mechanics, electromagnetism, optics, and acoustics). Many semiclassical methods have been carried out in the past to study such problems. A very powerful one is the geometrical theory of diffraction (GTD) developed by Keller [1] in order to describe the evolution of waves in terms of rays. Another useful method is the semiclassical trace formula introduced by Gützwiller [2, 3] and extended by other authors [4-7], using cycle expansions of zeta functions or quantum Fredholm determinants. Afterwards, the GTD has been incorporated by Vattay, Wirzba, Rosenqvist, and Whelan [8-10] in the Gützwiller trace formula in order to take account of the diffraction effects due to creeping waves. This periodic orbit theory of diffraction improves previous results, but errors still exist [9].

We propose here a semiclassical approach to extract and interpret all the scattering resonances of the two impenetrable cylinders scattering problem. The characteristic determinants of the scattering matrices involved in the problem are expanded in terms of simple traces which are evaluated using the Watson transformation [11]. Generalized formulas have been derived for all the contributions which are purely geometrical or composite, i.e., with a geometrical part (one or more reflections) and a diffractive part (creeping sections) (for more details, see Ref. [12]). The physical interpretation of the resonances is realized.

## I. SEMICLASSICAL THEORY

We consider two infinite, identical, impenetrable and parallel cylinders of radius $a$ with a center-to-center distance $d$. In previous papers [13-15], an exact formalism has been developed by emphasizing the role of the symmetries of the scatterer. The two-cylinder system has a $C_{2v}$ symmetry [16] with four one-dimensional irreducible representations labeled $A_1$, $A_2$, $B_1$, and $B_2$. We present here our method for the $A_1$ representation. The results are easily generalized to the three others representations of $C_{2v}$. The $A_1$ scattering resonances are the zeros in the complex $ka$-plane of the characteristic determinant (see Refs. [5, 15])

$$\det \mathbf{M}^{(A_1)} = 0 \quad \text{with } \mathbf{M}^{(A_1)} = \mathbf{I} + \mathbf{A}^{(A_1)}, \tag{1}$$

$$\mathbf{A}^{(A_1)}_{qp} = -\frac{\epsilon_p}{4}(-1)^q\left(S_q(ka)-1\right)\left[H^{(1)}_{p-q}(kd)+(-1)^p H^{(1)}_{p+q}(kd)\right] \tag{2}$$
$$\left(\epsilon_0 = 1, \epsilon_p = 2 \text{ for } p>0\right).$$

The vector $S_q(ka)$ includes the boundary conditions (b.c.) and is given for the Neumann b.c. by

$$S_q(ka) = -\frac{H_q^{(2)'}(ka)}{H_q^{(1)'}(ka)}.$$

From now on, to ease up the notation, the $A_1$-dependance is suppressed. We use the cumulant expansion given in Refs. [6, 12]. Introducing the notations

$$\varphi_q = \mathrm{Tr}\left(\mathbf{A}^q\right) \quad \text{for } q \geq 1, \tag{3}$$

the first two cumulants read

$$Q_1(\mathbf{A}) = \varphi_1, \quad Q_2(\mathbf{A}) = \frac{1}{2}\left[\varphi_2 - (\varphi_1)^2\right]. \tag{4}$$

In what follows, we extract all the periodic orbits from the first two terms of the cumulant expansion in a natural way using the Watson transformation [11], the method of steepest descent [17], the residue theorem [18], and high frequency approximations.

<u>A. The First Term Of The Cumulant Expansion</u>

The first order cumulant (4) is rewritten using Eqs. (1), (2) and (3). We apply the usual Watson transformation [11] to convert the partial wave series into a contour integral therefore

$$\varphi_1 = -\frac{i}{4}\int_C \frac{S_\nu(ka)-1}{\sin(\pi\nu)}\left[H_0^{(1)}(kd)+e^{i\pi\nu}H_{2\nu}^{(1)}(kd)\right]d\nu. \tag{5}$$

The contour $C$ encircles the real axis in the clockwise sense. It should be noted that the integration takes into account the Cauchy principle value at the origin. The deformation of the contour $C$ permits one to extract from Eq. (5) a purely diffractive contribution $\varphi_{dif,1}$ using the residue theorem and a purely geometrical contribution $\varphi_{g,1}$. Using Debye asymptotic expansions [19], the residue-series contribution $\varphi_{dif,1}$ reads

$$\varphi_{dif,1} = \varphi^{I}_{dif,1} + \varphi^{II}_{dif,1} \text{ with } \varphi^{I}_{dif,1} = -\sqrt{\frac{2i\pi}{kd}}\exp(ikd)\sum_{\nu_n} r(\nu_n)\frac{\exp(i\pi\nu_n)}{1-\exp(2i\pi\nu_n)},$$

$$\varphi^{II}_{dif,1} = -\sqrt{\frac{2i\pi}{k\sqrt{d^2-4a^2}}}\exp\left[ik\sqrt{d^2-4a^2}\right]\sum_{\nu_n} r(\nu_n)\frac{\exp\left[2i\nu_n\left(\pi-\arccos\frac{2a}{d}\right)\right]}{1-\exp(2i\pi\nu_n)}.$$

Here $\nu_n$ denotes the poles of the $S_\nu$ function in the complex $\nu$-plane (Refs. [12, 20]) and $r(\nu_n)$ is the residue of $S_\nu$ at the poles $\nu = \nu_n$. The purely geometrical contribution $\varphi_{g,1}$ is evaluated in the high frequency limit $ka \gg 1$ and $kd \gg 1$ using the method of steepest descent

$$\mathbf{f}_{g,1}=\mathbf{f}^{I}_{g,1}+\mathbf{f}^{II}_{g,1} \text{ with } \mathbf{f}^{I}_{g,1}=\frac{1}{2}R(0,ka)\sqrt{\frac{a}{2d}}\exp\left[ik(d-2a)\right],$$
$$\mathbf{f}^{II}_{g,1}=\frac{1}{2}R(0,ka)\sqrt{\frac{a}{2(d-2a)}}\exp\left[ik(d-2a)\right].$$

We have introduced the reflection coefficient $R(\nu, ka)$ which is defined according to the b.c. and its value is $-1$ for Neumann b.c. . Finally, the first cumulant is asymptotically approximated by

$$Q_1(\mathbf{A})=\mathbf{f}_{dif,1}+\mathbf{f}_{g,1}\ .$$

B. The Second Term Of The Cumulant Expansion

The second order cumulant is given by Eq. (4). The term $(\mathbf{f}_1)^2$ is directly deduced from the first order cumulant whereas we have to evaluate the term $\mathbf{f}_2$ given by Eq. (3). Using the expressions of the matrix elements (2), $\mathbf{f}_2$ reads

$$\mathbf{f}_2=\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\mathbf{A}_{pp}=\sum_{p=0}^{+\infty}\sum_{q=0}^{+\infty}\frac{g_q}{4}(-1)^p\left(s_p(ka)-1\right)\frac{g_p}{4}(-1)^q\left(s_q(ka)-1\right)X(p,q)$$
$$\text{with } X(p,q)=\left[H^{(1)}_{p-q}(kd)+(-1)^q H^{(1)}_{p+q}(kd)\right]\left[H^{(1)}_{q-p}(kd)+(-1)^p H^{(1)}_{q+p}(kd)\right].$$

Using the Watson transformation, we replace the two sums over the integers p, q by two contour integrals over the complex numbers $\nu_1$, $\nu_2$.

$$\mathbf{f}_2=-\frac{1}{16}\int_{C_1}\frac{s_{\nu_1}(ka)-1}{\sin(\pi\nu_1)}\left[\int_{C_2}\frac{s_{\nu_2}(ka)-1}{\sin(\pi\nu_2)}X(\nu_1,\nu_2)d\nu_2\right]d\nu_1. \quad (6)$$

The contours $C_1$, $C_2$ encircle the real positive axis in the clockwise sense in the corresponding complex $\nu_1$-plane and $\nu_2$-plane. In order to evaluate the double integral (6), we define

$$F(\nu_1)=\int_{C_2}\frac{s_{\nu_2}(ka)-1}{\sin(\pi\nu_2)}X(\nu_1,\nu_2)d\nu_2, \qquad \mathbf{f}_2=-\frac{1}{16}\int_{C_1}\frac{s_{\nu_1}(ka)-1}{\sin(\pi\nu_1)}F(\nu_1)d\nu_1.$$

We independently proceed to the modifications of the $C_2$ contour in the complex $\nu_2$-plane and of the $C_1$ contour in the complex $\nu_1$-plane, following the method used for the first order cumulant. We obtain a residue-series contribution and a geometrical contribution for each one

$$F(\nu_1)=F_g(\nu_1)+F_d(\nu_1), \qquad \mathbf{f}_2=\mathbf{f}_g\left[F(\nu_1)\right]+\mathbf{f}_d\left[F(\nu_1)\right].$$

Consequently, three different contributions are obtained for $\mathbf{f}_2$ (with simplified notations)

$$\mathbf{f}_2=\mathbf{f}_{dd,2}+2\mathbf{f}_{dif,2}+\mathbf{f}_{g,2}\,.$$

$\mathbf{f}_{dd,2}$ is a purely diffractive contribution deduced from previous results. $\mathbf{f}_{dif,2}$ is a composite contribution, i.e., it contains a diffractive part and a geometrical part evaluated by using the method of steepest descent. $\mathbf{f}_{g,2}$ is a purely geometrical contribution obtained applying twice the method of steepest descent on $\nu_1$ and $\nu_2$. After simplifications, the second cumulant reads

$$Q_2(\mathbf{A})=-\frac{1}{2}\left[\mathbf{f}_{g,2}-(\mathbf{f}_{g,1})^2\right]+\mathbf{f}_{dif,1}\mathbf{f}_{g,1}-\mathbf{f}_{dif,2} \quad \text{with} \begin{Bmatrix}\mathbf{f}_{dif,q}=\mathbf{f}^{I}_{dif,q}+\mathbf{f}^{II}_{dif,q}\\ \mathbf{f}_{g,q}=\mathbf{f}^{I}_{g,q}+\mathbf{f}^{II}_{g,q}\end{Bmatrix} \text{ for } q=1,2.$$

More detailed results are given in Ref. [12]. The same procedure is applied to the third order cumulant and a generalization is derived.

C. Generalization

Generalizing previous results concerning the cumulants, $\det\mathbf{M}$ for the $A_1$-representation of $C_{2v}$ can be semiclassically evaluated for any truncation order q by

$$\det\mathbf{M}=\sum_{q=0}^{+\infty}\left[Q_{g,q}+\sum_{m=1}^{q}(-1)^{m+1}Q_{g,q-m}\,\mathbf{f}_{dif,m}\right],$$

where $Q_{g,q}$ is defined as a geometrical cumulant

$$Q_{g,0} = 1, \quad Q_{g,q} = \frac{1}{q}\sum_{m=1}^{q}(-1)^{m+1} Q_{g,q-m}\, \mathbf{f}_{g,m} \quad \text{for } q \ge 1.$$

$\mathbf{f}_{\mathrm{dif},m}$ is the m-order composite contribution composed by one diffractive part and by (m-1) geometrical parts. $\mathbf{f}_{g,m}$ is the m-order geometrical contribution. The method described in case of the $A_1$-representation can be easily extended to the three other irreducible representations $A_2$, $B_1$ and $B_2$ of $C_{2v}$ using simple modifications (see Ref.[12]).

## II. NUMERICAL RESULTS AND PHYSICAL INTERPRETATION OF RESONANCES

The scattering resonances of the two-cylinder system are physically interpreted as periodic paths using the expressions obtained for the $A_1$ representation.

### A. Purely Geometrical Contributions

The geometrical contributions are of the form

$$\mathbf{f}_{g,q} \propto \exp\left[qik(d-2a)\right], \quad \text{for } q \ge 1,$$

where the exponential term provides the periodic orbit interpretation. These contributions are obviously associated with the closed geometrical path described in Fig. 1. More precisely, q corresponds to the number of reflections on the cylinder.

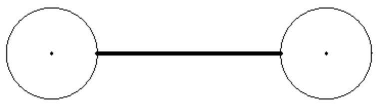

FIG. 1: Periodic orbit of the geometrical contributions $\mathbf{f}_{g,q}$.

### B. Composite Contributions

From the previous results, all the composite contributions are of the form

$$\mathbf{f}^{l}_{dif,q} \propto \exp(ikt)\exp(i\nu_n\beta), \quad \text{for } q \ge 1 \text{ with } l = I, II.$$

*t* denotes the geometrical path between the two cylinders and β stands for the angle of the creeping section. For instance, Figs. 2 and 3 display the periodic orbit deduced from the two first order diffractive contributions.

FIG. 2: 1st order periodic orbit $\mathbf{f}^{I}_{dif,1}$ with $t = d$, $\beta = \pi$.

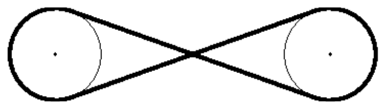

FIG. 3: 1st order periodic orbit $\mathbf{f}^{II}_{dif,1}$ with $t = \sqrt{d^2 - 4a^2}$, $\beta = 2\pi - 2\arccos(2a/d)$.

Figures 4, 5 display the periodic orbits deduced from the third order composite contributions. These periodic orbits present creeping sections around the cylinders and a number of reflections growing up with the order q of the composite contribution. In the limit of high q-value, the composite contributions go to a limit periodic path.

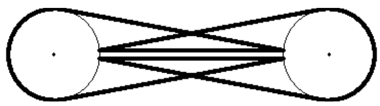

FIG. 4: 3rd order periodic orbit $\mathbf{f}^{I}_{dif,3}$.

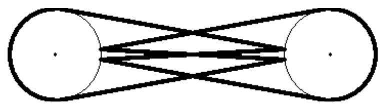

FIG. 5: 3rd order periodic orbit $\mathbf{f}^{II}_{dif,3}$.

### C. Exact versus asymptotic scattering resonances

We present here a comparison between the exact resonances and the asymptotic resonances calculated from our semiclassical theory for Neumann b.c. in the complex ka-plane for the center-to-center distance d=6a. The exact scattering resonances are the zeros of det**M** and have been determined in the restricted domain $0 \le \mathrm{Re}(ka) \le 50$ and $-1.8 \le \mathrm{Im}(ka) \le 0$ using the argument principle [21]. The exact resonances are compared to resonances obtained with our semiclassical approach for the first three cumulants in the case of Neumann b.c.. At the first, second, and third truncation orders, the expansion of $\det \mathbf{M}$ reads

$$\det \mathbf{M}_{(1)} = Q_0(\mathbf{A}) + Q_1(\mathbf{A}), \tag{7}$$

$$\det \mathbf{M}_{(2)} = Q_0(\mathbf{A}) + Q_1(\mathbf{A}) + Q_2(\mathbf{A}), \tag{8}$$

$$\det \mathbf{M}_{(3)} = Q_0(\mathbf{A}) + Q_1(\mathbf{A}) + Q_2(\mathbf{A}) + Q_3(\mathbf{A}), \tag{9}$$

where $Q_0(\mathbf{A})$, $Q_1(\mathbf{A})$, $Q_2(\mathbf{A})$ and $Q_3(\mathbf{A})$ are calculated with the asymptotic formulas. Figure 6 displays the comparison between exact and first order asymptotic resonances [the zeros of Eq. (7)]. We observe a good agreement for the resonances lying on the line close to the real $ka$-axis. They are associated with the first order geometrical contribution presented on Fig. 1. The first order approximation provides a second asymptotic line whose resonances do not match the exact ones. They are associated with the purely diffractive contribution $\mathbf{f}_{\mathrm{dif},1}$. It should also be noted that the second asymptotic line is located deeper inside the $ka$-plane and contains less resonances than the exact second lne. The first approximation of $\det \mathbf{M}$ does not provide the complete location of exact resonances in the studied region. We must therefore take into account the second cumulant.

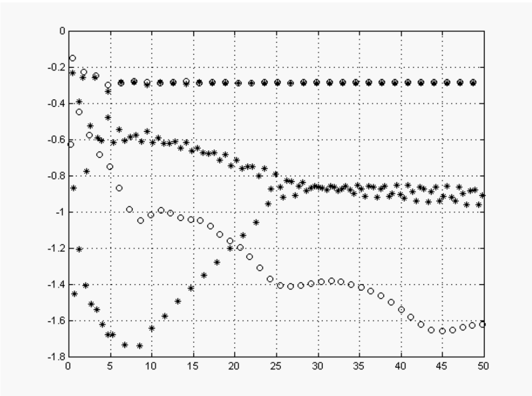


FIG. 6 : Exact resonances $(*)$ and 1st order asymptotic resonances $(\circ)$ in the $ka$-plane (Neumann b.c., $d = 6a$, $\mathrm{A}_1$ representation).

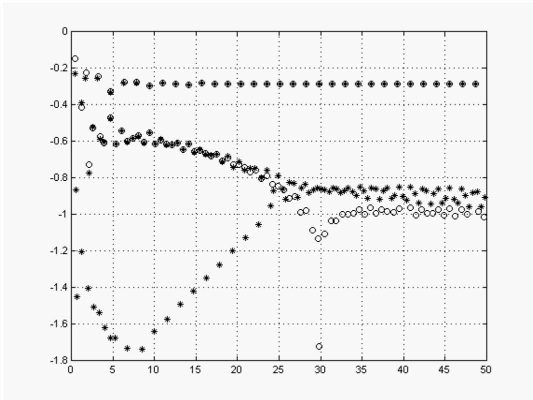


FIG. 7 : Exact resonances $(*)$ and 2nd order asymptotic resonances $(\circ)$ in the $ka$-plane (Neumann b.c., $d = 6a$, $\mathrm{A}_1$ representation).

Figure 7 displays the comparison between exact and second order asymptotic resonances [the zeros of Eq. (8)]. The first asymptotic line still match the exact data. The second resonances line is well approximated up to $\mathrm{Re}(ka) > 25$. The corresponding asymptotic resonances are associated with the diffractive contributions $\mathbf{f}_{\mathrm{dif},1}$, $\mathbf{f}_{\mathrm{dif},2}$ and with the second order geometrical contribution $\mathbf{f}_{g,2}$. A third exact line is not displayed by the second order expansion of $\det \mathbf{M}$, we therefore take into account the third order cumulant.

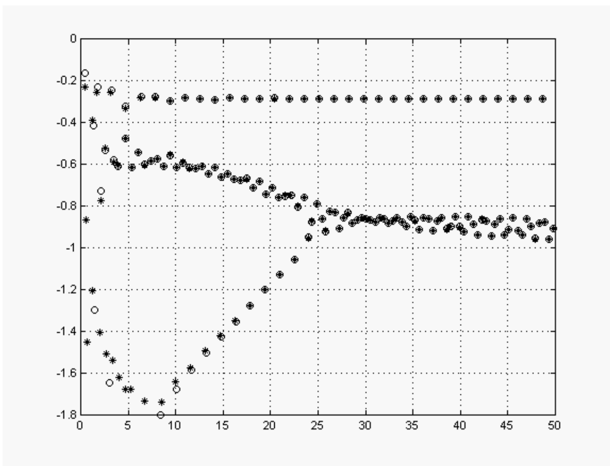


FIG. 8 : Exact resonances $(*)$ and 3rd order asymptotic resonances $(\circ)$ in the $ka$-plane (Neumann b.c., $d = 6a$, $\mathrm{A}_1$ representation).

Figure 8 displays the comparison between exact and third order asymptotic resonances [the zeros of Eq. (9)]. A very good agreement is obtained in the whole studied domain. Nevertheless a weak discrepancy is observed in the region Re(ka)d 8 and Im(ka)d-1.2 where the asymptotic expansions used are not very efficient in this region. The third line, coming from Re(ka) > 8, Im(ka)> -1.7 and joining the second line near Re(ka) > 25, Im(ka)> -0.9, is associated with the third order geometrical contribution $\mathbf{f}_{g,3}$ and with the composite contribution $\mathbf{f}_{dif,3}$. Similar results are obtained for the three other representations $A_2$, $B_1$, $B_2$ of the $C_{2v}$ symmetry group in cases of Dirichlet and impedance b.c. (see Ref. [12]).

## III. CONCLUSION

The two impenetrable cylinders scattering problem has been entirely solved. All the scattering resonances of the cumulant expansion have been extracted and interpreted in terms of periodic orbits. We have obtained a semiclassical approximation of the characteristic determinant for each irreducible representation of the $C_{2v}$ symmetry group. Moreover, our semiclassical approach provides scattering resonances in excellent agreement with the exact results. We can then postulate that scattering of waves and particles by two identical, impenetrable cylinders is a canonical problem.

The semiclassical formalism developed in this paper is actually extended to the more difficult scattering problem by two penetrable cylinders (fluid b.c. in acoustics, or mixed b.c. in quantum physics). It should be noted that multiple scattering problems by penetrable objects have never been semiclassically treated.

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