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ANALYSIS OF SOUND PROPAGATION OVER ABSORBING GROUND

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INTRODUCTION

The analytical task of spherical-wave propagation over an absorbing plane started by SOMMERFELD's paper 80 years ago, and his main problem, i.e. the existence or non-existence of surface waves and their importance, is still alive. Numerous papers have appeared since then on this topic. They mostly deal with approximate numerical solutions to the exact integral representations of the field patterns.A pecularity of the literature lies in the fact, that approximate results are mostly compared against each others. The numerical integration of the exact solution is said to be too complicated to be performed. It is hard, on this basis, to judge on the relative merits of different approximations.Exact numerical results are presented here, therefore. This is important, because most of the "higher order approximations" of the literature really give no improvement, or even are worse than the "simple approximations", a further pecularity of the literature.

EXACT SOLUTIONS

The time factor be $\exp(-i\omega t)$, as usual in literature; the geometry is depicted in **Fig.1**. A point source is placed in S , a receiver in $P(r,z)$. The plane $z=0$ separates the upper medium with the characteristic wave number and field impedance k_1 , Z_1 from the lower medium, which is either bulk reacting, then with the characteristic wave number and field impedance k_2 , Z_2 , or which is locally reacting, then with the normalized (with Z_1) wall impedance Z. The sound pressure in the upper medium be $p_1(r,z)$, or if we like to indicate the source height h explicitely $p_1(r,z;h)$, and we normalize by an appropriate pressure P_0 so that the free spherical wave of the point source reads $p_1(r,z)/P_0 = \exp(ik_1R_1)/(k_1R_1)$.

In the case of a *bulk reacting half-space* $z<0$, using $k=k_2/k_1$ and $Z=Z_2/Z_1$, and in the special situation of source height $h=0$ ($R_1=R_2=R$) (SOMMERFELD's situation), the solution is:

$$\frac{p_1(r,z;0)}{P_0} = 2(1+kZ)\int_0^\infty \frac{y\,J_0(yk_1r)\,e^{-k_1z\sqrt{y^2-1}}}{\sqrt{y^2-k^2}+kZ\sqrt{y^2-1}}\,dy =: 2(1+kZ)\cdot I \qquad (1)$$

The generalization to non-zero source heights, $h\neq 0$, according to BREKHOVSKIKH, then is.

$$\frac{p_1(r,z;h)}{P_0} = \frac{e^{ik_1R_1}}{k_1R_1} - \frac{e^{ik_1R_2}}{k_1R_2} + \frac{p_1(r,z+h;0)}{P_0} \qquad (2)$$

Eq.(1) is based on a decomposition of spherical waves into cylindrical ones, [1]. By a decomposition into plane waves which then are reflected by the plane-wave reflection factor $r(\Theta)$, one gets:

$$\frac{p_1}{P_0} = \frac{e^{ik_1R_1}}{k_1R_1} + \frac{p_r}{P_0}$$

with:

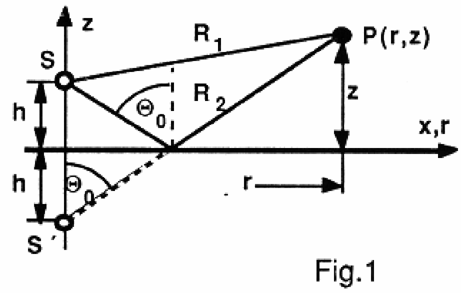

Fig.1

$$\frac{p_r}{P_0} = i\int_0^{\pi/2} J_0(k_1 r\sin\Theta')e^{ik_1 H\cos\Theta'}\cdot r(\Theta')\cdot\sin\Theta'\,d\Theta' + \tag{3}$$

$$+\int_0^{\infty} J_0(k_1 r\cosh\Theta'')e^{-k_1 H\sinh\Theta''}\cdot r(\tfrac{\pi}{2}-i\Theta'')\cdot\cosh\Theta''\,d\Theta'' =: i\,I_1 + I_2$$

with $H=h+z$ and with the reflection factors (4)

$$r(\Theta') = \frac{kZ\cos\Theta' - \sqrt{k^2-1+\cos^2\Theta'}}{kZ\cos\Theta' + \sqrt{k^2-1+\cos^2\Theta'}}\,;\; r(\tfrac{\pi}{2}-i\Theta'') = \frac{kZ\tanh\Theta'' - \sqrt{1-(k/\cosh\Theta'')^2}}{kZ\tanh\Theta'' + \sqrt{1-(k/\cosh\Theta'')^2}}$$

Eq.(3) is suited for *locally reacting planes*, also, because the absorber is represented by the reflection factor only, which now is:

$$r(\Theta') = \frac{Z\cos\Theta' - 1}{Z\cos\Theta' + 1}\;;\; r(\tfrac{\pi}{2}-i\Theta'') = \frac{Z\sinh\Theta'' + i}{Z\sinh\Theta'' - i} \tag{5}$$

NUMERICAL INTEGRATIONS

A numerical integration starts from the integrals I_1, I_2 in eq.(3), which, however, are not suited for numerical integration as they stand due to the strong variation of the "periods" of the Bessel functions. They first are modified by a transformation $y=\cos\Theta'$ in I_1 and $y=\sinh\Theta''$ in I_2 which leads to (for a *bulk reacting half-space*):

$$I_1 = \int_0^1 J_0(k_1 r\sqrt{1-y^2})\frac{kZ\,y - \sqrt{k^2-1+y^2}}{kZ\,y + \sqrt{k^2-1+y^2}}e^{+ik_1 H\,y}\,dy \tag{6}$$

$$I_2 = \int_0^{\infty} J_0(k_1 r\sqrt{1+y^2})\frac{kZ\,y - \sqrt{1+y^2-k^2}}{kZ\,y + \sqrt{1+y^2-k^2}}e^{-ik_1 H\,y}\,dy$$

If k_1H is small (or even zero), then the integrand in I_2 becomes small for $y\to\infty$ only by the Bessel function $J_0(x)$, i.e. as $1/\sqrt{x}$ which is very slow. The same holds for I in eq.(1) (where we replace z by H according to eq.(2)). We therefore first apply an acceleration of convergence.

For this we write I in eq.(1) as $I(a(y),b(y),c(y))$ and I_2 in eq.(6) as $I_2(A(y),B(y),C(y))$ with the factors and their asymptotic approximations for large y:

$$a(y) = J_0(y\,k_1 r) \to a_\infty(y) = \sqrt{\frac{2}{\pi k_1 r y}}\cos(y\,k_1 r - \pi/4) \tag{7}$$

$$b(y) = \frac{y}{\sqrt{y^2-k^2} + kZ\sqrt{y^2-1}} \to b_\infty = \frac{1}{1+kZ}\;;\; c(y) = e^{-k_1 H\sqrt{y^2-1}} \to c_\infty = e^{-k_1 H y}$$

and

$$A(y) = J_0(k_1 r\sqrt{1+y^2}) \to A_\infty = J_0(y\,k_1 r)$$

$$B(y) = \frac{kZy - \sqrt{1+y^2-k^2}}{kZy + \sqrt{1+y^2-k^2}} \to B_\infty = \frac{kZ-1}{kZ+1}\;;\; C(y) = C_\infty(y) = e^{-k_1 H y} \tag{8}$$

Then we evidently have:
$$I(a,b,c) = I(a,b_\infty,c_\infty) + \int_0^{\infty} a\cdot(bc - b_\infty c_\infty)\,dy \tag{9}$$

and with
$$I(a,b_\infty,c_\infty) = \frac{1}{1+kZ}\frac{1}{k_1 R_2} \tag{10}$$

one gets for I in eq.(1):

$$I(a,b,c) = \frac{1}{1+kZ}\frac{1}{k_1 R_2} + \int_0^{\infty} J_0(yk_1 r)\left[\frac{y\,e^{-k_1 H\sqrt{y^2-1}}}{\sqrt{y^2-k^2} + kZ\sqrt{y^2-1}} - \frac{e^{-k_1 H y}}{1+kZ}\right]dy \tag{11}$$

Here the integrand decreases as $(1/y^2)\cdot\exp(-k_1 H y)$, which means that the upper integration limit must not be extended so far.

Similarly we write for I_2 in eq.(6): $I_2(A,B,C) = I_2(A_\infty, B_\infty, C) + \int_0^\infty (AB - A_\infty B_\infty) C\, dy$ (12)

and get:

$$I_2 = \frac{kZ-1}{kZ+1}\frac{1}{k_1 R_2} + \int_0^\infty \left[J_0(k_1 r\sqrt{1+y^2})\frac{kZy - \sqrt{1+y^2-k^2}}{kZy + \sqrt{1+y^2-k^2}} - J_0(k_1 ry)\frac{kZ-1}{kZ+1}\right] e^{-k_1 Hy}\, dy \tag{13}$$

Although this looks more complicated than eq.(11), it has the advantage, that the Bessel function is included in the acceleration of convergence with only slow oscillations of the integrand for large y which allows for a reduction of the step number in the Romberg integration scheme. The computing time is reduced by this acceleration of convergence by a factor of about 1/4 to 1/5 compared to immediate integration, and the precision altogether is improved.

For a *locally reacting plane* $z=0$, we start from eq.(2), now with (14)

$$\frac{p_1(r, z+h; 0)}{P_0} = 2Z\int_0^\infty J_0(yk_1 r)\frac{y}{Z\sqrt{y^2-1} - i} e^{-k_1 H\sqrt{y^2-1}}\, dy$$

which, by the way, follows from eq.(1) in the limit $k \to \infty$ and Z the normalized wall impedance.

With the integral written as in eq.(9) and the factors and their asymptotics:

$$a = a_\infty = J_0(yk_1 r)\ ;\ b = \frac{y}{Z\sqrt{y^2-1} - i} \to b_\infty = \frac{1}{Z}\ ;\ c = e^{-k_1 H\sqrt{y^2-1}} \to c_\infty = e^{-k_1 Hy} \tag{15}$$

one gets:

$$\frac{p_1(r, H; 0)}{P_0} = \frac{2}{k_1 R_2} + 2\int_0^1 J_0(yk_1 r)\left[i \cdot \frac{y\, e^{ik_1 H\sqrt{1-y^2}}}{\sqrt{1-y^2} + 1/Z} - e^{-k_1 Hy}\right] dy + 2\int_1^\infty J_0(yk_1 r)\left[\frac{y\, e^{-k_1 H\sqrt{y^2-1}}}{\sqrt{y^2-1} - i/Z} - e^{-k_1 Hy}\right] dy \tag{16}$$

This can be integrated quite easily by a Romberg scheme with an increase of the upper integration limit of the second integral by steps until the variation due to such an increase remains under a certain limit.

EXACT PASS INTEGRATION

Most of the approximations existing in the literature are derived by application of the pass integration (steepest-descent integration) as a tool for approximate integration, and most of the controverse discussion concerning the surface waves comes from the question whether or not a pole contribution thereby will appear (see below). We here apply an exact pass integration, because numerical integration along the path of steepest descent (pass way) is attractive numerically, also.

A starting point is eq.(3), where, after replacement of the Bessel function by the Hankel function and multiplication by $\exp(\pm ik_1 r \cdot \sin\vartheta)$ one gets:

$$\frac{p_r}{P_0} = \frac{i}{2}\int_{C_2} e^{ik_1 R_2\cos(\vartheta - \Theta_0)} H_0^{(1)}(k_1 r\sin\vartheta) e^{-ik_1 r\sin\vartheta} \cdot r(\vartheta)\cdot \sin\vartheta\, d\vartheta = \frac{i}{2} I_0 \tag{17}$$

with the product of the 2nd and 3rd factor under the integral a rather monotonous function for large arguments, and the path of integration C_2: $-\pi/2+i\infty \to -\pi/2 \to +\pi/2 \to +\pi/2-i\infty$.

The integral I_0 is of the type $\int_C e^{x\cdot f(\vartheta)} F(\vartheta)\, d\vartheta$ (18)

which is appropriate for pass integration if x (in our case $k_1 R_2$) is a large real number and $F(\vartheta)$ is a sufficiently steady function. The integration path is transformed in the complex plane $\vartheta = \vartheta' + i\cdot\vartheta''$ to the pass way P which passes through the saddle point ϑ_S at the place of the maximum magnitude of the exponential factor, which is determined from $df(\vartheta)/d\vartheta = 0$, which in our case is $\vartheta_S = \Theta_0$. The pass way is determined from the requirement of constant phase, i.e. from $\mathrm{Im}\, f(\vartheta_P) = const = \mathrm{Im}\, f(\vartheta_S)$ (index P means "on the pass way"). Its equation in our case is:

$$\cos(\vartheta'_P - \Theta_0)\cosh\vartheta''_P = 1 \tag{19}$$

which, with the addition theoreme for trigonometric functions, can be solved for:

$$\sin\vartheta'_p = \frac{\sin\Theta_0 - \cos\Theta_0 \sinh\vartheta''_p}{\cosh\vartheta''_p} \; ; \cos\vartheta'_p = \frac{\cos\Theta_0 + \sin\Theta_0 \sinh\vartheta''_p}{\cosh\vartheta''_p} \tag{20}$$

So the function $f(\vartheta)$ in eq.(18) becomes on the pass way:

$$f(\vartheta_p) = -\tanh\vartheta''_p \sinh\vartheta''_p + i \xrightarrow[\vartheta''_p \to 0]{} -(\vartheta''_p)^2 + i \tag{21}$$

The saddle point is at $\vartheta_p''=0$. The shape of the pass way is constant, it is shifted towards the right side with increasing Θ_0.

All functions of ϑ in eq.(17) - more precisely of ϑ_p on the pass way - now can be expressed as functions of ϑ_p''. For example (for a *locally reacting plane*) : (22)

$$r(\vartheta_p) = \frac{Z \cdot [\cos\Theta_0 + \sin\Theta_0 \sinh\vartheta''_p - i\tanh\vartheta''_p \cdot (\sin\Theta_0 - \cos\Theta_0 \sinh\vartheta''_p)] - 1}{Z \cdot [\ldots \qquad \ldots] + 1} =: r\langle\vartheta''_p\rangle$$

and:

$$\sin\vartheta_p = \sin\Theta_0 - \cos\Theta_0 \sinh\vartheta''_p + i\tanh\vartheta''_p \cdot (\cos\Theta_0 + \sin\Theta_0 \sinh\vartheta''_p) =: \sin\langle\vartheta''_p\rangle$$
$$\cos\vartheta_p = \cos\Theta_0 + \sin\Theta_0 \sinh\vartheta''_p - i\tanh\vartheta''_p \cdot (\sin\Theta_0 - \cos\Theta_0 \sinh\vartheta''_p) =: \cos\langle\vartheta''_p\rangle \tag{23}$$

The integral I_0 finally becomes (if we put $\varphi=\vartheta_p''$ for ease of writing):

$$I_0 = e^{ik_1R_2} \int_0^\infty e^{-k_1R_2 \tanh\varphi \sinh\varphi} \frac{2-\tanh^2\varphi}{i + 1/\cosh\varphi} [r\langle\varphi\rangle \sin\langle\varphi\rangle H_0^{(1)}(k_1 r \sin\langle\varphi\rangle) e^{-ik_1 r \sin\langle\varphi\rangle} +$$
$$+ r\langle-\varphi\rangle \sin\langle-\varphi\rangle H_0^{(1)}(k_1 r \sin\langle-\varphi\rangle) e^{-ik_1 r \sin\langle-\varphi\rangle}] \, d\varphi \tag{24}$$

The terms in the brackets are rather monotonous, and the exponential decreases rapidly with increasing φ, So the number of steps in numeric integrations - and thereby the computing time - is reduced to about 1/10 of the number needed for the former integrations. Nevertheless, the computation is exact under the following condition.

POLE CONTRIBUTION

Eqs.(3),(21) and (24) will give the complete solution if no singularity of the integrand has been crossed when transforming the integration path C_2 to the pass way P . With a *bulk reacting absorber* there are branch cuts of the roots in the reflection factor r of eq.(4). They, however, produce no problems, because the pass way does not cross over to the other river side of these cuts. Next, the reflection factor has poles at the zeroes of its denominator. There are four zeroes, two of which are at the same time zeroes of the nominator, so they produce no poles. Of the two others, only the pole ϑ_p with $\vartheta_p'>\pi/2$ and $\vartheta_p''<0$ is of interest. It can be shown (see [1]) that for *bulk reacting absorbers* it always lays outside the pass way, so no pole contribution is necessary in this case.

For *locally reacting absorbers*, however, the pass way P encompasses the pole position under the condition (see [1])

$$(1/Z)'' < -\frac{1}{\sin\Theta_0} \frac{[\cos\Theta_0 + (1/Z)'][1 + \cos\Theta_0 \cdot (1/Z)']}{\sqrt{1 + 2\cos\Theta_0 \cdot (1/Z)' + (1/Z)'^2}} \tag{25}$$

Then the pole contribution, which must be added to p_r/P_0 follows from the residuum theoreme:

$$\frac{p_{rp}}{P_0} = -\frac{2\pi}{Z} H_0^{(1)}(k_1 r \sqrt{1 - 1/Z^2}) e^{-k_1 H/Z} \; ; \; \mathrm{Re}\sqrt{1 - 1/Z^2} > 0 \tag{26}$$

Since $(1/Z)''<0$, p_{rp}/P_0 decreases exponentially with increasing H; it is a surface wave. The condition (25) can be shown (see [1]) to be just the condition for the existence of free surface waves along a locally reacting absorber plane ! It must be stated, that the condition is satisfied very often for absorbing ground, if it is modelled as a locally reacting absorber.

[1] MECHEL, F.P. "Schallabsorber",Vol.1,Ch.13, Hirzel Verl.,Suttgart,1989