The physics of tibetan singing bowls

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Abstract

Tibetan bowls have been traditionally used for ceremonial and meditation purposes, but are also increasingly being used in contemporary music-making. They are handcrafted using alloys of several metals and produce different tones, depending on the alloy composition, their shape, size and weight. Most important is the sound producing technique used – either impacting or rubbing, or both simultaneously – as well as the excitation location, the hardness and friction characteristics of the exciting stick (called *puja*).

In this paper, we extend to axi-symmetrical shells – subjected to impact and friction-induced excitations – our modal techniques of physical modelling, which were already used in previous papers concerning plucked and bowed strings as well as impacted and bowed bars. Our simulation results highlight the existence of several motion regimes, both steady and unsteady, with either permanent or intermittent bowl/*puja* contact. Furthermore, the unstable modes spin at the angular velocity of the puja. As a consequence, for the listener, singing bowls behave as rotating quadropoles. The sound will always be perceived as beating phenomena, even if using perfectly symmetrical bowls.

Introduction

Singing bowls are traditionally made in Tibet, Nepal, India, China and Japan. Although the name *qing* has been applied to lithophones since the Han Chinese Confucian rituals, more recently it also designates the bowls used in Buddhist temples. There are many distinct bowls, which produce different tones, depending on the alloy composition, their shape, size and weight (see three typical specimens in Figure 1). Most important is the sound producing technique used – either impacting or rubbing, or both simultaneously – as well as the excitation location, the hardness and friction characteristics of the exciting stick (called *puja*, frequently made of wood and eventually covered with a soft skin).

Quite recently, some researchers became interested in the physical modelling of singing bowls, using waveguide synthesis techniques for performing numerical simulations [1-3]. Their efforts aimed particularly at achieving real-time synthesis. Therefore, understandably, several aspects of the physics of these instruments do not appear to be clarified in the published formulations and results.

In this paper, we extend to axi-symmetrical shells – subjected to impact and friction-induced excitations – our modal techniques of physical modelling, which were already used in previous papers concerning plucked and bowed strings [4-6] as well as impacted and bowed bars [7,8]. Our approach is based on a modal representation of the unconstrained system – here consisting on two orthogonal families of modes of similar (or near-similar) frequencies and shapes. The bowl modeshapes have radial and tangential motion components, which are prone to be excited respectively by the normal and frictional contact forces between the bowl and the impact/sliding *puja*.

Details on the specificities of the contact and frictional models used in our simulations are given. We produce an extensive series of nonlinear numerical simulations, for both impacted and rubbed bowls, showing the influence of the contact/friction parameters on the dynamical responses.

Figure 1 – Picture of three singing bowls and pujas: Bowl 1 (ϕ=180 mm); Bowl 2 (ϕ=152 mm); Bowl 3 (ϕ=140 mm).

Formulation of the dynamical system

Dynamical Formulation of the Bowl in Modal Coordinates

Perfectly axi-symmetrical structures exhibit double vibrational modes, occurring in orthogonal pairs with identical frequencies ($\omega_n^A = \omega_n^B$) [9]. However, if a slight alteration of this symmetry is introduced, the natural frequencies of these two degenerate modal families deviate from identical values by a certain amount $\Delta\omega_n$. The use of these modal pairs is essential for the correct dynamical description of axi-symmetric bodies, under general excitation conditions. Furthermore, shell modeshapes present both radial and tangential components which for geometrically perfect bowls can be formulated, at the rim level Z, as:

$$\varphi_n^A(\theta) = \varphi_n^{Ar}(\theta)\vec{r} + \varphi_n^{At}(\theta)\vec{t} \quad \text{and} \quad \varphi_n^B(\theta) = \varphi_n^{Br}(\theta)\vec{r} + \varphi_n^{Bt}(\theta)\vec{t} \qquad (3,4)$$

$$\text{with} \quad \begin{cases} \varphi_n^{Ar}(\theta) = \cos(n\theta) \\ \varphi_n^{At}(\theta) = -\sin(n\theta)/n \end{cases} \quad ; \quad \begin{cases} \varphi_n^{Br}(\theta) = \sin(n\theta) \\ \varphi_n^{Bt}(\theta) = \cos(n\theta)/n \end{cases} \qquad (5,6)$$

where $\varphi_n^{Ar}(\theta)$ corresponds to the radial component of the *A* family *n*th modeshape, $\varphi_n^{At}(\theta)$ to the tangential component of the A family *n*th mode shape, etc. One immediate conclusion can be drawn from equations (5,6): the amplitude of the tangential modal component decreases relatively to the amplitude of the radial component as the mode number increases. This suggests that only the lower-order modes are prone to engage in self-sustained motion due to tangential rubbing excitation by the *puja*.

If linear dissipation is assumed, the motion of the system can be described in terms of the bowl's two families of modal parameters: modal masses m_n^X, modal circular frequencies ω_n^X, modal damping ζ_n^X, and mode shapes $\varphi_n^X(\theta)$ (at the assumed excitation level $z_e \approx Z$), with n=1,2,..., N, where X stands for the modal family A or B. The order N of the modal truncation is problem-dependent and should be asserted by physical reasoning, supported by the convergence of computational results. The maximum modal frequency to be included, ω_N, mostly depends on the short time-scales induced by the contact parameters – all modes significantly excited by impact and/or friction phenomena should be included in the computational modal basis.

The forced response of the damped bowl can then be formulated as a set of 2N ordinary second-order differential equations:

$$\begin{bmatrix} [M_A] & 0 \\ 0 & [M_B] \end{bmatrix} \begin{Bmatrix} \{\ddot{Q}_A(t)\} \\ \{\ddot{Q}_B(t)\} \end{Bmatrix} + \begin{bmatrix} [C_A] & 0 \\ 0 & [C_B] \end{bmatrix} \begin{Bmatrix} \{\dot{Q}_A(t)\} \\ \{\dot{Q}_B(t)\} \end{Bmatrix} + \begin{bmatrix} [K_A] & 0 \\ 0 & [K_B] \end{bmatrix} \begin{Bmatrix} \{Q_A(t)\} \\ \{Q_B(t)\} \end{Bmatrix} = \begin{Bmatrix} \{\Xi_A(t)\} \\ \{\Xi_B(t)\} \end{Bmatrix} \qquad (7)$$

where:

$[M_X] = \text{Diag}(m_1^X, \cdots, m_N^X)$, $[C_X] = \text{Diag}(2 m_1^X \omega_1^X \zeta_1^X, \cdots, 2 m_N^X \omega_N^X \zeta_N^X)$,

$[K_X] = \text{Diag}(m_1^X (\omega_1^X)^2, \cdots, m_N^X (\omega_N^X)^2)$ are the matrices of the modal parameters (where X stands for A or B), for each of the two orthogonal mode families, while $\{Q_X(t)\} = \langle q_1^X(t), \cdots, q_N^X(t)\rangle^T$ and $\{\Xi_X(t)\} = \langle \Im_1^X(t), \cdots, \Im_N^X(t)\rangle^T$ are the vectors of the modal responses and of the generalized forces, respectively. Note that, although equations (7) obviously pertain to a linear formulation, nothing prevents us from including in $\Im_n^X(t)$ all the nonlinear effects which arise from the contact/friction interaction between the bowl and the *puja*. Accordingly, the system modes become coupled by such nonlinear effects.

The modal forces $\Im_n^X(t)$ are obtained by projecting the external force field on the modal basis:

$$\Im_n^X(t) = \int_0^{2\pi} \left[F_r(\theta,t)\, \varphi_n^{Xr}(\theta) + F_t(\theta,t)\, \varphi_n^{Xt}(\theta) \right] d\theta \quad ; \quad n = 1, 2, \cdots, N \qquad (8)$$

where $F_r(\theta,t)$ and $F_t(\theta,t)$ are the radial (impact) and tangential (friction) force fields applied by the *puja* – e.g., a localised impact $F_r(\theta_c,t)$ and/or a travelling rub, $F_{r,t}(\theta_c(t),t)$. The radial and tangential physical motions can be then computed at any location θ from the modal amplitudes $q_n^X(t)$ by superposition:

$$y_r(t) = \sum_{n=1}^{N} \left[\varphi_n^{Ar}(\theta) \cdot q_n^A(t) + \varphi_n^{Br}(\theta) \cdot q_n^B(t) \right] \quad ; \qquad (9)$$

$$y_t(t) = \sum_{n=1}^{N} \left[\varphi_n^{At}(\theta) \cdot q_n^{A}(t) + \varphi_n^{Bt}(\theta) \cdot q_n^{B}(t) \right] \qquad (10)$$

and similarly concerning the velocities and accelerations.

Dynamics of the Puja and Force Field Formulation

As mentioned before, the excitation of these musical instruments can be performed in two basic different ways: by impact or by rubbing around the rim of the bowl with the *puja* (these two types of excitation can obviously be mixed, resulting in musically interesting effects). The dynamics of the *puja* will be formulated simply in terms of a mass m_P subjected to a normal (e.g. radial) force $F_N(t)$ and an imposed tangential rubbing velocity $V_T(t)$ – which will be assumed constant in time for all our exploratory simulations – as well as to an initial impact velocity in the radial direction 0 $V_N(t_0)$. These three parameters will be assumed controlled by the musician, and many distinct sounds may be obtained by changing them: in particular, V_N=T_0≠0 with F_N=V_T=0 will be "pure" impact, and $F_N(t)$≠, $V_T(t)$≠ with $V_N(t_0)$=0 will be "pure" singing. The radial motion of the *puja*, resulting from the external force applied and the impact/friction interaction with the bowl is given by:

$$m_P \ddot{y}_P = -F_N(t) + F_r(\theta, t) \qquad (11)$$

Contact Interaction Formulation

The radial contact force resulting from the interaction between the puja and the bowl is simply modelled as a contact stiffness, eventually associated with a contact damping term:

$$F_r(\theta_c) = -K_c\, \tilde{y}_r(\theta_c, t) - C_c\, \dot{\tilde{y}}_r(\theta_c, t) \qquad (12)$$

where $\tilde{y}_r$ and $\dot{\tilde{y}}_r$ are respectively the bowl/puja relative radial displacement and velocity, at the (fixed or travelling) contact location $\theta_c(t)$, K_c and C_c are the contact stiffness and damping coefficients, directly related to the puja material and local geometry.

Friction Interaction Formulation

In previous papers we have shown the effectiveness of a friction model used for the simulation of bowed strings and bowed bars [4,7]. Such model enabled a clear distinction between sliding and adherence states, sliding friction forces being computed from the Coulomb mode $F_t = -|F_r|\, \mu_d(\dot{\tilde{y}}_t)\mathrm{sgn}(\dot{\tilde{y}}_t)$, where $\dot{\tilde{y}}_t$ is the bowl/*puja* relative tangential velocity, and the adherence state being modelled essentially in terms of a local "adherence" stiffness K_a and some damping. We were thus able to emulate true friction sticking of the contacting surfaces, whenever $|F_t| < |F_r|\mu_s$, however at the expense of a longer computational time, as smaller integration time-steps seem to be imposed by the transitions from velocity-controlled sliding states to displacement-controlled adherence states.

In this paper, a simpler approach is taken to model friction interaction, which allows for faster computation times, although it lacks the capability to emulate true friction sticking. The friction force is here formulated as:

$$\begin{cases} F_t(\theta_c, t) = -\left|F_r(\theta_c, t)\right| \mu_d\left(\dot{\tilde{y}}_t(\theta_c, t)\right) sgn\left(\dot{\tilde{y}}_t(\theta_c, t)\right), & \text{if } \left|\dot{\tilde{y}}_t(\theta_c, t)\right| \geq \varepsilon \\ F_t(\theta_c, t) = -\left|F_r(\theta_c, t)\right| \mu_s\, \dot{\tilde{y}}_t(\theta_c, t) / \varepsilon & \text{, if } \left|\dot{\tilde{y}}_t(\theta_c, t)\right| < \varepsilon \end{cases} \qquad (13)$$

where μ_s is a "static" friction coefficient and $\mu_d(\dot{\tilde{y}}_t)$ is a "dynamic" friction coefficient, which depends on the puja/bowl relative surface velocity $\dot{\tilde{y}}_t$. We will use the following model:

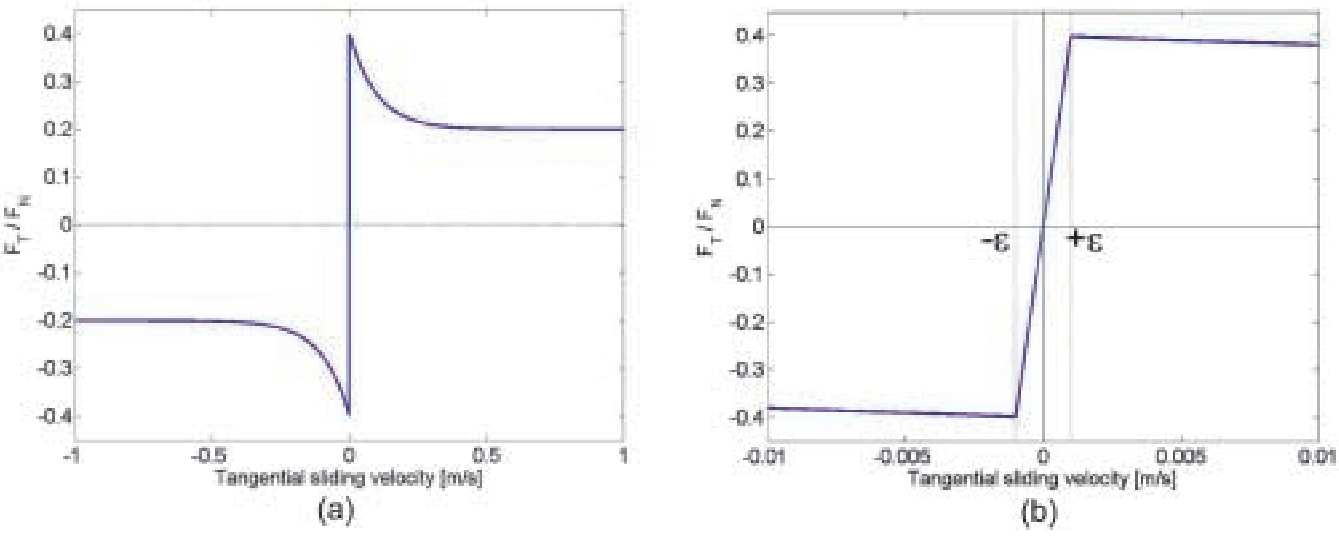

Figure 2 – Evolution of the friction coefficient with the contact relative tangential velocity (μ_∞ = 0.2, μ_s = 0.4, C = 10): (a) For -1< $\dot{\tilde{y}}_t$ <1; (b) For -0.01< $\dot{\tilde{y}}_t$ <0.01

Mode (j,k)[1]	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)	(7,0)	(8,0)
f_n $[Hz]$	314	836	1519	2360	3341	4462	5696
f_n / f_1	1,0	2,7	4,8	7,5	10,7	14,2	18,2

Table 1 – Modal frequencies and frequency ratios of the Bowl 2 (m_T = 563 g, ϕ = 152 mm)

$$\mu_d(\dot{\tilde{y}}_t) = \mu_\infty + (\mu_s - \mu_\infty)\exp\left(-C\left|\dot{\tilde{y}}_t(\theta_c, t)\right|\right) \tag{14}$$

where, $0 \le \mu_\infty \le \mu_s$ is an asymptotic lower limit of the friction coefficient when $|\dot{\tilde{y}}_t| \to \infty$, and parameter C controls the decay rate of the friction coefficient with the relative sliding velocity, as shown in the typical plot of Figure 2(a). This model can be fitted to the available experimental friction data (obtained under the assumption of instantaneous velocity-dependence), by adjusting the empirical constants μ_s , μ_∞ and C . Obviously, ε acts as a regularization parameter for the friction force law, which replaces the "zero-velocity" discontinuity [13]. For the problem addressed here, we have obtained realistic results using formulation (13), for small enough values of the regularization domain (we used $\pm\varepsilon \approx 10^{-4}\,ms^{-1}$) – results which do not seem to critically depend on ε (within reasonable limits).

Time-Step Integration

For given external excitation and initial conditions, the previous system of equations is numerically integrated using an adequate time-step algorithm. Explicit integration methods are well suited for the contact/friction model developed here. In our implementation, we used the simple Velocity-Verlet integration algorithm, which is a low-order explicit scheme [14].

Numerical simulations

The numerical simulations presented here are based on the modal data of Bowl 2 shown in Figure 1 (with rim diameter of 152 mm, a total mass of 563 g and a fundamental frequency of 314 Hz), which were experimentally identified (see Table 1). The *puja* is modeled as a simple mass of 20 g, moving at tangential velocity V_T , and subjected to an external normal force F_N as well as to the bowl/*puja* nonlinear interaction force.

We explored a significant range of rubbing parameters: F_N = 1 ~ 9 N and V_T = 0.1 ~ 0.5 m/s, which encompass the usual playing conditions, although calculations were made also using impact excitation only. For clarity, the normal force and tangential velocity were assumed time-constant, in the present simulations. The contact model used in all rubbing simulations has a contact stiffness of $K_c = 10^6$ N/m and a contact dissipation of C_c = 50 Ns/m, which appear adequate for

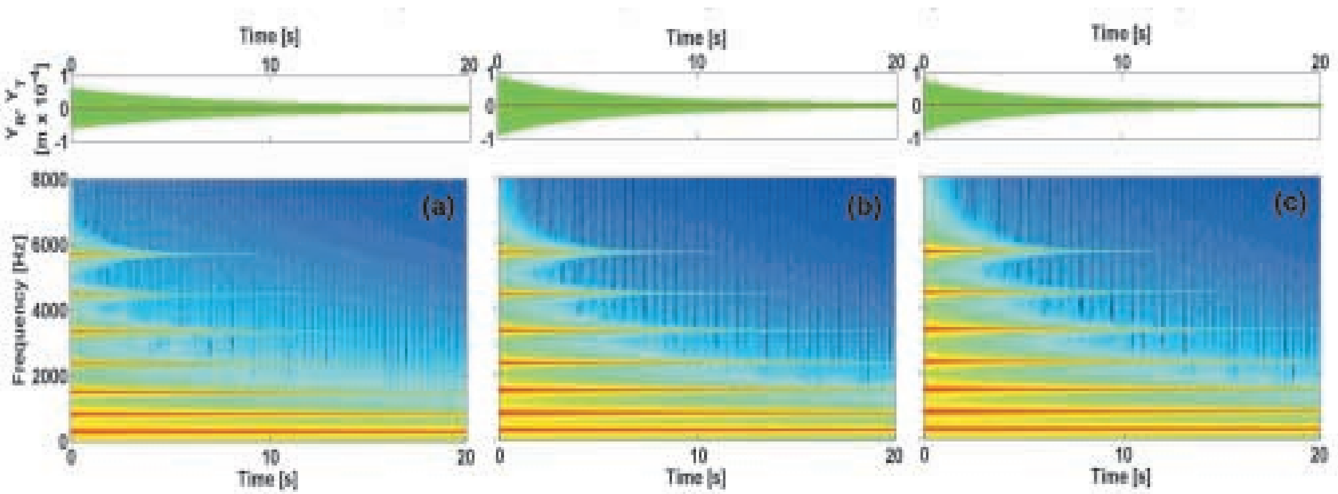

Figure 3 – Displacement time histories (top) and spectrograms (bottom) of the response of Bowl 2 to impact excitation with different values of the bowl/puja contact stiffness: (a) 10^5 N/m ; (b) 10^6 N/m ; (c) 10^7 N/m

[1] Notation (j,k) represents the number of complete nodal meridians extending over the top of the bowl (half the number of nodes observed along a circumference), and the number of nodal circles, respectively [12].

the present system. However, concerning impact simulations, contact parameters ten times higher and lower were also explored. The friction parameters used in numerical simulations of rubbed bowls are $\mu_S = 0.4$, $\mu_D = 0.2$ and $C = 10$. No effort, at this stage, was made to explore other friction laws, however the parameters used tentatively here seem realistic enough.

Seven mode pairs were used to describe the dynamics of Bowl 2 (see Table 1 and Figure 4 in [9]). An average value of 0.005% was used for all modal damping coefficients. Assuming a perfectly symmetrical bowl, simulations were performed using identical frequencies for each mode-pair ($\omega_n^A = \omega_n^B$). In order to cope with the large settling times that arise with singing bowls, 20 seconds of computed data were generated (enough to accommodate transients for all rubbing conditions), at a sampling frequency of 22050 Hz.

Impact Responses

Figures 3(a-c) display the simulated responses of a perfectly symmetrical bowl to an impact excitation ($V_N(t_0) = 1$ m/s), assuming different values for the contact model parameters. The time-histories of the response displacements pertain to the impact location. The spectrograms are based on the corresponding velocity responses. Typically, as the contact stiffness increases from 10^5 N/m to 10^7 N/m, higher-order modes become increasingly excited and resonate longer. The corresponding simulated sounds become progressively brighter, denoting the "metallic" bell-like tone which is clearly heard when impacting real bowls using wood or metal *pujas*.

Friction-Excited Responses

Figure 4 shows the results obtained when rubbing a perfectly symmetrical bowl near the rim, using fairly standard rubbing conditions: $F_N = 3$ N and $V_T = 0.3$ m/s. The plots shown pertain to the following response locations: (a) the travelling contact point between the bowl and the *puja*; (b) a fixed point in the bowl's rim. Depicted are the time-histories and corresponding spectra of the radial (green) and tangential (red) displacement responses, as well as the spectrograms of the radial velocity responses.

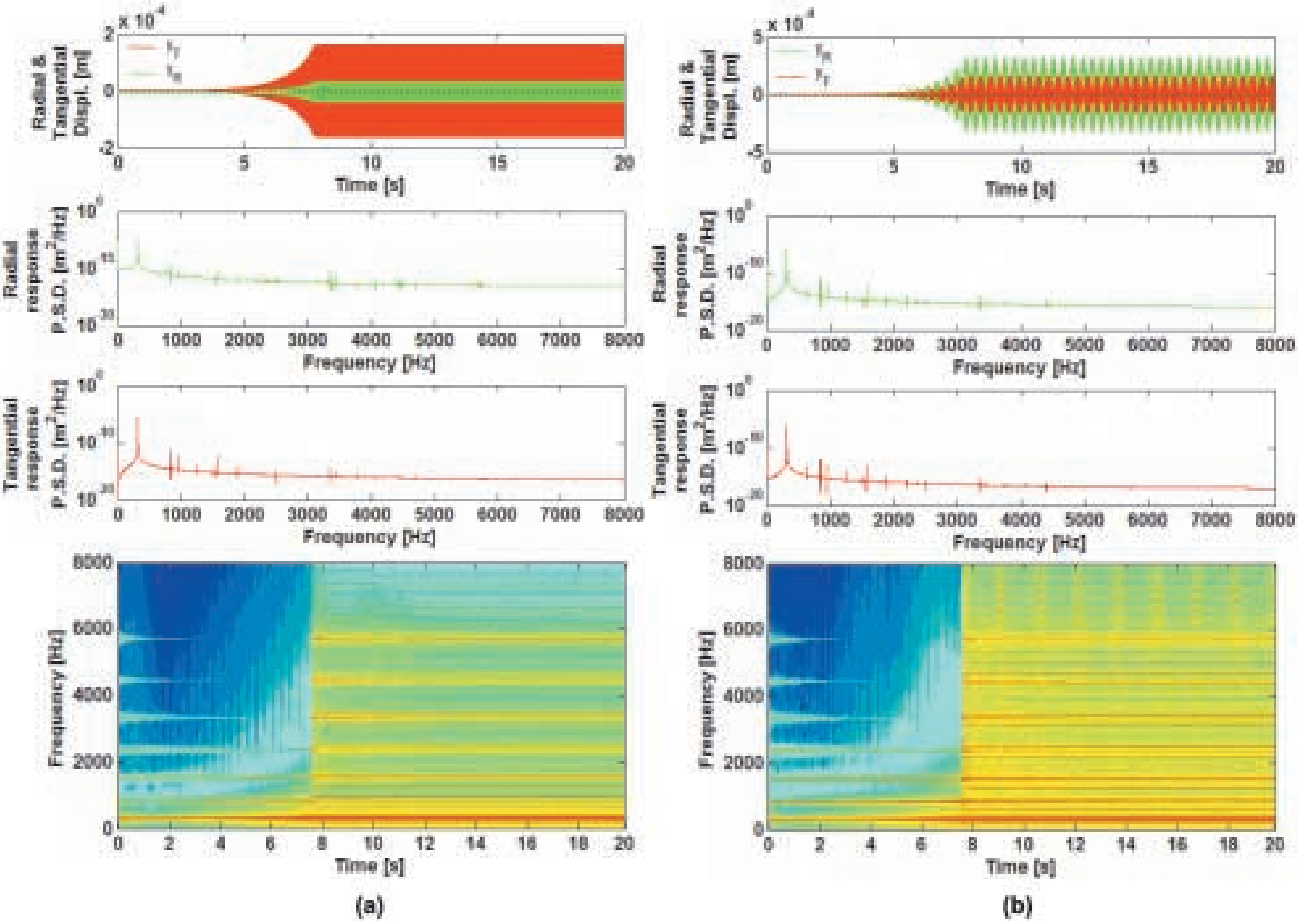

Figure 4 – Time-histories, spectra and spectrograms of the dynamical response of Bowl 2 to friction excitation when F_N = 3 N, V_T = 0.3 m/s: (a) at the bowl/puja travelling contact point; (b) at a fixed point of the bowl's rim

As can be seen, an instability of the first "elastic" shell mode (with 4 azimuthal nodes) arises, with an exponential increase of the vibration amplitude until saturation by nonlinear effects is reached (at about 7.5 s), after which the self-excited vibratory motion of the bowl becomes steady. The response spectra show that most of the energy lays in the first mode, the others being marginally excited. Notice the dramatic differences between the responses at the travelling contact point and at a fixed location. At the moving contact point, the motion amplitude does not fluctuate and the tangential component of the motion is significantly higher than the radial component. On the contrary, at a fixed location, the motion amplitude fluctuates at a frequency which can be identified as being four times the spinning frequency of the *puja*: $\Omega_{fluct} = 4\Omega_{puja} = 4(2V_T/\phi$. Furthermore, at a fixed location, the amplitude of the radial motion component is higher than the tangential component.

The animations of the bowl and *puja* motions enable an interpretation of these results. After synchronisation of the self-excited regime, the combined responses of the first mode-pair result in a vibratory motion according to the 4-node modeshape, which however spins, “following” the revolving *puja*. Furthermore, synchronisation settles with the puja interacting near a node of the radial component, corresponding to an anti-nodal region of the *tangential* component – see Equations (5,6). In retrospect, this appears to make sense – indeed, because of the friction excitation mechanism in singing bowls, the system modes self-organize in such way that a high *tangential* motion-component will arise at the contact point, where energy is inputted.

At any fixed location, a transducer will “see” the vibratory response of the bowl modulated in amplitude, as the 2*j* alternate nodal and anti-nodal regions of the “singing” modeshape revolve. For a listener, the rubbed bowl behaves as a spinning quadropole – or, in general, a 2*j*-pole (depending on the self-excited mode *j*) – and the radiated sound will always be perceived with beating phenomena, even for a perfectly symmetrical bowl.

It should be noted that our results basically support the qualitative remarks provided by Rossing, when discussing friction-excited musical glass-instruments (see [12], pp. 185-187 – the only reference, to our knowledge, where some attention has been paid to these issues). However, his main point *“The location of the maximum motion follows the moving finger around the glass”* may

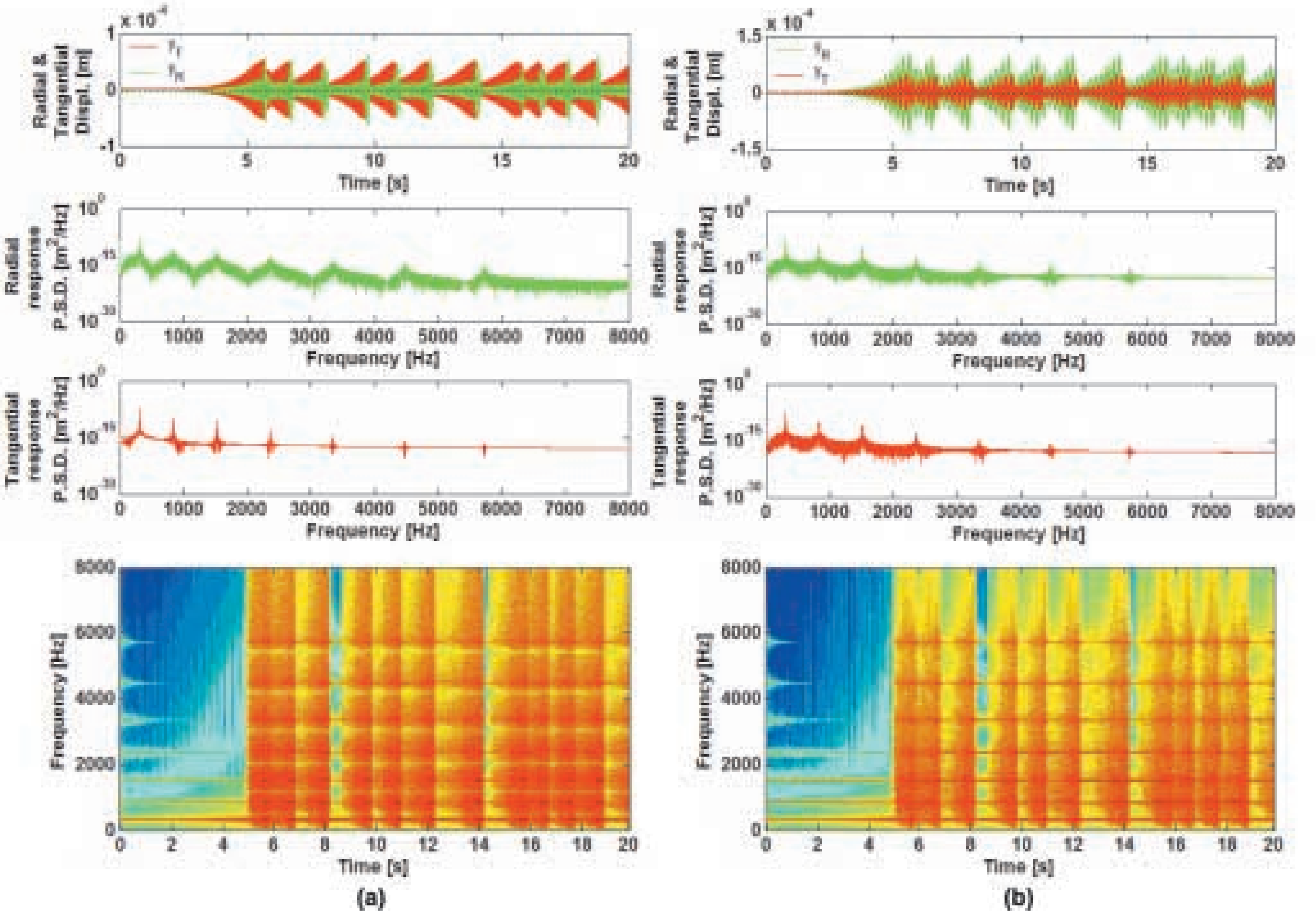

Figure 5 – Time-histories, spectra and spectrograms of the dynamical response of Bowl 2 to friction excitation when $F_N = 1$ N, $V_T = 0.5$ m/s: (a) at the bowl/puja travelling contact point; (b) at a fixed point of the bowl’s rim

now be further clarified: the "maximum motion" following the exciter should refer in fact to the maximum *tangential* motion component (and not the radial component, as might be assumed).

Figure 5 shows a quite different behaviour, when $F_N = 1$ N and $V_T = 0.5$ m/s. Here, a steady motion is never reached, as the bowl/*puja* contact is disrupted whenever the vibration amplitude reaches a certain level. At this point, severe chaotic impacting arises which breaks the mechanism of energy transfer, leading to a sudden decrease of the motion amplitude. Then, the motion build-up starts again until the saturation level is reached, and so on. As can be expected, this intermittent response regime results in curious sounds, which interplay the aerial characteristics of "singing" with a distinct "ringing" response due to chaotic chattering.

Conclusions

In this paper we have presented a modelling technique based on a modal approach which can achieve accurate time-domain simulations of impacted and/or rubbed axi-symmetrical structures such as the Tibetan singing bowl.

The numerical simulations presented show some light on the sound-producing mechanisms of Tibetan singing bowls. Both impact and friction excitations have been addressed. For suitable friction parameters and for adequate ranges of the normal contact force F_N and tangential rubbing velocity V_T of the *puja*, instability of a shell mode (typically the first "elastic" mode, with 4 azimuthal nodes) arises, with an exponential increase of the vibration amplitude followed by saturation due to nonlinear effects.

Because of the intimate coupling between the radial and tangential shell motions, the effective bowl/*puja* contact force is not constant, but oscillates. After vibratory motions settle, the excitation point tends to locate near a nodal region of the *radial* motion of the unstable mode, which corresponds to an anti-nodal region of the friction-excited *tangential* motion. This means that unstable modes spin at the same angular velocity of the *puja*. As a consequence, for the listener, sounds will always be perceived with beating phenomena. However, for a perfectly symmetrical bowl, no beating at all is generated at the moving excitation point.

The first motion regime offers the "purest" bowl singing. Our results suggest that higher values of F_N should enable a better control of the produced sounds, as they lead to shorter transients and also render the system less prone to chattering.

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