Effective properties derivation of Willis-type 1D asymmetric resonant structures

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Abstract
Bi-isotropic media and their acoustic analogues, Willis materials, have attracted considerable interest in recent years. These materials have the particular feature of coupling potential energy and kinetic energy in their constitutive equations, thus enhancing the control and design possibilities for targeted applications. In this work, we derive a general analytical methodology to access the effective and coupling parameters from the transfer matrix together with the Pade's approximation. We then analyze the derived closed-forms for three different one-dimensional asymmetric and reciprocal unit cells composed of two detuned Helmholtz resonators, two plates, or one Helmholtz resonator and one plate. The closed forms are analyzed to study the influence of the nature of the asymmetry on the coupling terms and are compared with experimental results.

Keywords: Willis metamaterials, Closed forms derivation, Asymmetric resonant structures.

1 Introduction

Bi-isotropic media [1] and their acoustic counterparts, Willis materials [2–4], have drawn considerable interest. These materials are unique in that they couple potential and kinetic energy in their constitutive equations, thereby improving control and design possibilities for desired applications. In this talk, we will derive and analyze the closed forms of the effective and coupling parameters for three different asymmetric and reciprocal unit cells consisting of two detuned Helmholtz resonators, two plates, or one Helmholtz resonator and one plate. Closed forms will be analyzed to study the influence of the nature of the asymmetry on the coupling terms.

2 Derivation of the closed-forms

We consider asymmetric and reciprocal one-dimensional media composed of a periodic arrangement of unit cells of thickness d . The effective parameter closed forms are derived from the unit cell transfer matrix $\mathbf{T}$ which relates the state vector $\mathbf{W} = \langle p, v\rangle^t$ (pressure p and particle velocity v) across the unit cell

$$\mathbf{W}(\mathbf{d}) = \mathbf{T}\mathbf{W}(0) = e^{\mathbf{A}d}\mathbf{W}(0). \quad (1)$$

The $\mathbf{A}$ matrix arises from the constitutive equations and reads for a Willis material as

$$\mathbf{A} = i\omega \begin{bmatrix} \chi & \rho \\ 1/K & -\chi \end{bmatrix}, \quad (2)$$

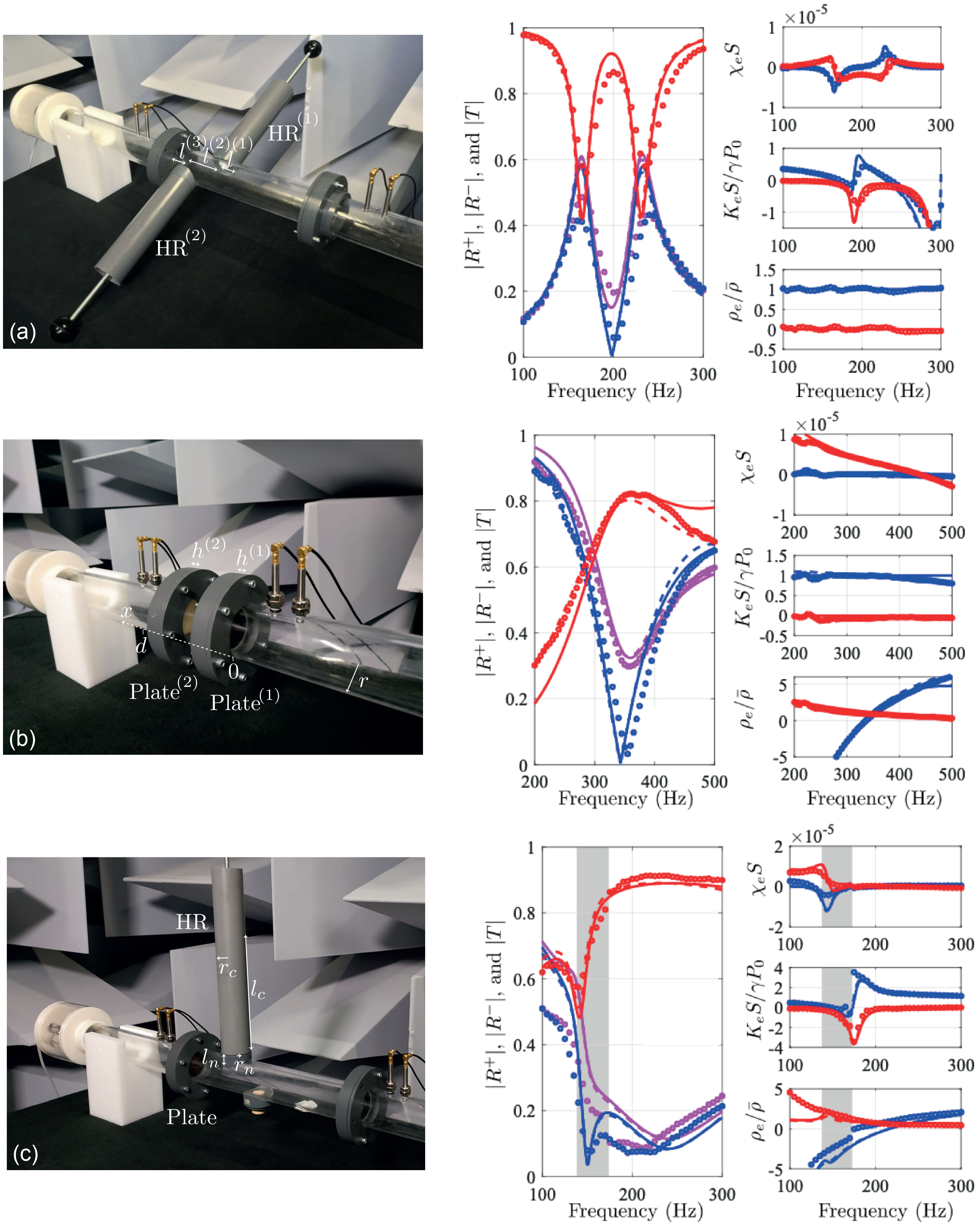

Figure 1: Photographs and results for the three cases analyzed: (a) two detuned Helmholtz resonators, (b) two detuned plates, and (c) plate and Helmoltz resonator. The different subplots show respectively the scattering magnitudes, the Willis coupling parameter, the normalized effective bulk modulus, and the normalized effective density, for each case. The solid lines represent the analytical, the dashed lines the numerical, and the symbols the experimental results.

where K the bulk modulus, ρ is the density, and χ the Willis coupling term.

Making use of the Padé's approximation of the matrix exponential,

$$\mathbf{T} = e^{\mathbf{A}d} \approx (\mathbf{Id} - \mathbf{A}d/2)^{-1} (\mathbf{Id} + \mathbf{A}d/2)\,, \tag{3}$$

one can approximate the matrix $\mathbf{A}$ in terms of the t_{ij} elements of the $\mathbf{T}$ matrix

$$\mathbf{A} \approx \frac{2}{d(2 + t_{11} + t_{22})} \begin{bmatrix} t_{11} - t_{22} & 2t_{12} \\ 2t_{21} & t_{22} - t_{11} \end{bmatrix}, \tag{4}$$

by ensuring the reciprocity of the transformation.

We can then identify the closed-forms of the effective parameters and Willis coupling term

$$\chi = \frac{-2i(t_{11} - t_{22})}{\omega d(2 + t_{11} + t_{22})}, \tag{5}$$

$$\rho = \frac{-4it_{12}}{\omega d(2 + t_{11} + t_{22})}, \tag{6}$$

$$[K]^{-1} = \frac{-4it_{21}}{\omega d(2 + t_{11} + t_{22})}. \tag{7}$$

3 Experimental validation of the derived closed-forms

The closed forms of the effective parameters for the three unit cells shown in Fig. 1 are compared with numerical simulations and experimental measurements. The measurements are performed with a 4-microphone impedance tube of radius $r = 2.5$ cm excited by a step-sine.

A very good agreement is found between the three methods, validating the analytical derivation of the effective parameters and the Willis coupling.

4 Conclusion

This work introduces a new approach to the comprehension of Willis metamaterials: the derivation of the effective parameters and the Willis coupling terms closed-forms via the Padé's approximation of the transfer matrix of different asymmetric and reciprocal one-dimensional unit cells. We discuss the different types of coupling terms depending on the nature of the asymmetry, either related to a detuning of identical type resonators or due to a physical asymmetry when the unit cell has different types of resonators.

A double validation is performed with numerical and experimental results that are in good agreement with the derived closed forms.

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