Fundamental constraints on broadband passive acoustic treatments

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Abstract
We apply the theory of Herglotz functions to derive fundamental constraints on broadband passive acoustic treatments in a unidimensional scattering problem. This reveals that the weighted integral of the reflection, transmission, or absorption spectrum is related to the total length of the treatment as well as to the static limit of its acoustic response. Since the static limit could be well predicted with a priori estimation on the structural filling ratio, this analysis makes it possible to evaluate the inherent constraints on the required minimum length of a passive acoustic treatment before any specific design.

Keywords: fundamental constraints, passive acoustic treatment, sum rules, Herglotz function.

1 Introduction

Passivity is an inherent property satisfied by many physical systems. In particular, for a continuous, causal, linear, and time-translational invariant system, passivity leads to sum rules and imposes fundamental constraints on the system, which relate the broadband response to low/high frequency behaviours. In order to derive these constraints, the Kramers-Kronig (K-K) relations [1, 2] and the Bode gain-phase relation (or modified K-K relation) [3, 4] are commonly adopted. Examples include the derivations of fundamental constraints or physical limitations on the electrical networks [5, 6], the absorbers of electromagnetic wave [7] or acoustic wave [8], the feedback control systems and filters [9], Compton scattering and vector bosons in nuclear physics [10], etc. Notice that, some further assumptions are usually made on the considered systems [11, 12], e.g., the response function is restricted to be rational so that the Cauchy integral formula is applicable [12]. To consider more general cases, the method developed by Bernland *et al.* [12] could be employed. This method is based on the theory of Herglotz function [11, 13] and has contributed to a wide variety of applications in electromagnetism [14, 15, 16].

Passive treatments are also widely used to achieve various functionalities in the applications of acoustic metamaterials, e.g., in a scattering problem, the design of broadband absorbers [17, 18, 19], silencers [20, 21], meta-diffusers [22], etc., and see [23, 24, 25] with the references therein for other applications. It could be deduced that, in many cases the concerned acoustic response cannot be expressed by a rational function (e.g., the acoustic impedance of a commonly used quarter-wavelength resonator is not a rational function). Thus, we apply the method of Ref.[12] in acoustics, to analyse the fundamental constraints on passive treatments. In Sec.2.1, this method as well as the theory of Herglotz functions is briefly introduced. The transfer matrix modelling of the considered scattering problem is provided in Sec.2.2 and the derived sum rules and fundamental constraints are discussed for a symmetric reflection problem, anti-symmetric reflection problem, and symmetric transmission problem in Sec.2.3 to Sec.2.5, respectively. Conclusions are drawn in Sec.3.

2 Fundamental constraints for a 1D scattering problem

2.1. Sum rules and fundamental constraints for a passive system

For a continuous, causal, linear, and time-translational invariant system, the output of the system is expressed as a convolution between the input and the response. After the time-domain Fourier transform, the convolution reduces to a product in the frequency domain and the response function is defined by the ratio between the output and input spectra. When the system is passive, the response function should construct a positive-real function [16]. Specifically, when the surface impedance $\zeta(\omega)$ is considered, the passivity of the system is equivalent to $\mathrm{Re}[\zeta(\omega)] \geq 0$, which ensures that the acoustic energy transferred into the system would not be amplified. On the other hand, if we consider the reflection and transmission coefficients, $R(\omega)$ and $T(\omega)$, in the scattering problem, the passivity of the system requires that $|R(\omega)|^2+|T(\omega)|^2 \leq 1$, i.e., $-\ln[|R(\omega)|^2+|T(\omega)|^2] \geq 0$, or separately, $-\ln|R(\omega)| \geq 0$ and $-\ln|T(\omega)| \geq 0$. These relations imply that the total output energy of the system should not be greater than the input energy.

The Herglotz function could be directly used to analyse the response of a passive system, since it is closely related to the positive-real function. With the time dependence of $\mathrm{e}^{-\mathrm{i}\omega t}$, a Herglotz function $H(\omega)$ is defined by a holomorphic function in the upper half complex ω plane, and $\mathrm{Im}[H(\omega)] \geq 0$ for $\mathrm{Im}(\omega) > 0$ [12, 13]. It follows that in the aforementioned passive acoustic systems, the Herglotz functions $H_1(\omega) = \mathrm{i}\zeta(\omega)$ and $H_2(\omega) = -\mathrm{i}\log[R(\omega)\cdot B(\omega)]$ (or $-\mathrm{i}\log[T(\omega)\cdot B(\omega)]$) could be introduced, where $\log(\cdot)$ is the complex logarithm, $B(\omega)$ is the Blaschke product [7, 12] to remove the zeros of $R(\omega)$ or $T(\omega)$ in the upper half plane. With the asymptotic series expansions of the Herglotz functions at both the static and dynamic limits, a series of integral identities or sum rules could be derived [12, 15]. These sum rules reveal fundamental constraints on the passive system.

2.2. Transfer matrix modelling

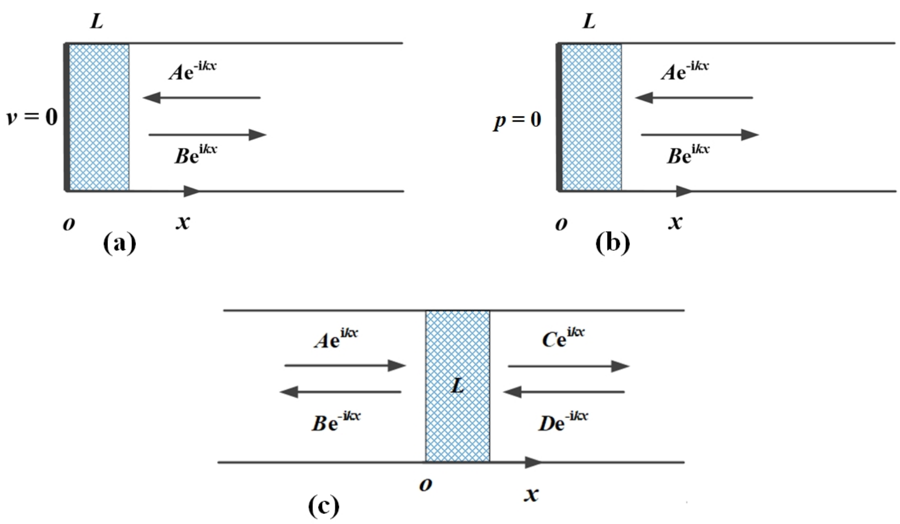

Figure 1: The 1D scattering problem of a plane wave by a composite material: (a) Symmetric reflection problem; (b) Anti-symmetric reflection problem; (c) Symmetric transmission problem.

As shown in Fig.1, we consider the one-dimensional (1D) scattering problem in which a plane incident wave is scattered by a composite material with thickness L. It is assumed that the material is composed by subwavelength structures, e.g., tubes, cavities etc., and the medium of the acoustic wave is air in the entire system. The losses of the system are induced by viscothermal boundary layers near the no-slip and isothermal boundaries. In the case that the acoustic performance of the system is symmetric in the axial direction, the composite material could be modelled as an equivalent fluid-like layer with frequency-dependent effective parameters. Under the above assumptions, the transfer matrix of the system is written by

$$\left[\begin{array}{c} P(L) \\ V(L) \end{array}\right] = \mathbf{T}\cdot\left[\begin{array}{c} P(0) \\ V(0) \end{array}\right] = \left[\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array}\right]\cdot\left[\begin{array}{c} P(0) \\ V(0) \end{array}\right], \tag{1}$$

where $P = p/K_0$, $V = v/c_0$ are the dimensionless acoustic pressure and particle velocity with $K_0 = \rho_0 c_0^2$ referring to the bulk modulus of the air, the elements of the matrix are $t_{11} = t_{22} = \cos(k_e L)$, $t_{12} = \mathrm{i}\frac{\rho_e c_e}{\rho_0 c_0}\sin(k_e L)$ and $t_{21} = \mathrm{i}\frac{\rho_0 c_0}{\rho_e c_e}\sin(k_e L)$. The equivalent density ρ_e, sound speed c_e and wavenumber k_e depend on the frequency, whose static/dynamic limits could be evaluated under the assumptions we have made for the material [26]. In either a reflection problem or a transmission problem, the acoustic response of the material could be expressed by the elements of the transfer matrix.

2.3. Symmetric reflection problem

As shown in Fig.1(a), consider a symmetric reflection problem in which a rigid boundary is set at one end of the material ($x = 0$). The surface impedance and reflection coefficient at $x = L$ are given by $\zeta(\omega) = -t_{21}/t_{11}$ and $R(\omega) = (t_{11} + t_{21})/(t_{11} - t_{21})$. With the help of the Herglotz functions H_1 and H_2, the fundamental constraints are derived for ζ and R, i.e.,

$$\frac{c_0}{2\pi}\frac{K_e(0)}{K_0}\int_0^\infty \frac{1}{\omega^2}\mathrm{Re}[\zeta(\omega)]\mathrm{d}\omega = L\,, \tag{2}$$

and

$$\frac{c_0}{\pi}\frac{K_e(0)}{K_0}\left|\int_0^\infty \frac{1}{\omega^2}\ln|R(\omega)|\,\mathrm{d}\omega\right| \le L\,, \tag{3}$$

respectively, where $K_e(0)$ is the static limit of the effective bulk modulus $K_e = \rho_e c_e^2$. Generally, $K_e(0)/K_0$ is evaluated by $1/(\sigma\gamma)$, where $\gamma = 1.4$ is the adiabatic index of the air and σ is the filling ratio of the composite material. Note that Eq.(3) is in accordance with the results given in Refs.[8, 17], which could be rearranged to a constraint on the absorption spectrum by using $\alpha(\omega) = 1 - |R(\omega)|^2$.

2.4. Anti-symmetric reflection problem

In the anti-symmetric reflection problem, where a pressure-release boundary is satisfied at $x = 0$ (Fig.1(b)), the concerned response functions are given by $\zeta(\omega) = -t_{12}/t_{22}$ and $R(\omega) = (t_{12} + t_{22})/(t_{12} - t_{22})$. When we consider the case that the material has a filling ratio σ close to 100%, the dimensionless quantity $L\nu_0/(\sigma c_0 D^2)$ which dominates the static/dynamic limits of ζ and R, is guaranteed to be much less than unity in common cases, with ν_0 the kinematic viscosity of air and D a typical length scale of the cross-section of the sub-wavelength structures used to compose the material. It follows that

$$\frac{c_0}{2\pi}\frac{\rho_0}{\mathrm{Re}[\rho_e(0)]}\int_0^\infty \frac{1}{\omega^2}\mathrm{Re}[\zeta(\omega)]\mathrm{d}\omega = L\,, \tag{4}$$

and

$$\frac{c_0}{\pi}\frac{\rho_0}{\mathrm{Re}[\rho_e(0)]}\left|\int_0^\infty \frac{1}{\omega^2}\ln|R(\omega)|\,\mathrm{d}\omega\right| \le L\,. \tag{5}$$

Note that, $\rho_e(0)$ depends on the cross-section geometry for each components and $\mathrm{Re}[\rho_e(0)]/\rho_0$ generally varies from $1.2/\sigma$ to $1.44/\sigma$ (see, p.64 of Ref.[26]). Similarly to the previous case, the inequality on $|R(\omega)|$ provides a constraint on $\alpha(\omega)$.

2.5. Symmetric transmission problem

In the symmetric transmission problem (Fig.1(c)), it is found that the static-limit sum rules are available merely for the transmission coefficient $T(\omega) = 2e^{\mathrm{i}k_0 L}/(t_{11} - t_{12} - t_{21} + t_{22})$, provided that $L\nu_0/(\sigma c_0 D^2) \ll 1$.

These sum rules are given by

$$\frac{2c_0}{\pi}\frac{1}{\dfrac{K_0}{K_\mathrm{e}(0)}+\dfrac{\mathrm{Re}[\rho_\mathrm{e}(0)]}{\rho_0}-2}\int_0^\infty \frac{1}{\omega^2}\mathrm{Im}\left[\mathrm{i}\frac{1-T(\omega)}{1+T(\omega)}\right]\mathrm{d}\omega = L\,, \tag{6}$$

and

$$\frac{4c_0}{\pi}\frac{1}{\dfrac{K_0}{K_\mathrm{e}(0)}+\dfrac{\mathrm{Re}[\rho_\mathrm{e}(0)]}{\rho_0}-2}\left|\int_0^\infty \frac{1}{\omega^2}\ln|T(\omega)|\,\mathrm{d}\omega\right| \le L\,. \tag{7}$$

Note that in the above identity (Eq.(6)), a bilinear transformation (given in p.131 of Ref.[27]) is used to construct a Herglotz function from $T(\omega)$. In this case the integrand does not refer to a surface impedance of the system. On the contrary, the inequality (Eq.(7)) could be used to evaluate bounds for the transmission loss as well as the absorption spectrum. In order to derive the latter, the relation $1-\alpha(\omega) = |T(\omega)|^2 + |R(\omega)|^2 \ge |T(\omega)|^2$ should be adopted.

3 Conclusion

In conclusion, we apply the theory of Herglotz function to derive sum rules for a 1D scattering problem in acoustic metamaterial applications, which relates the weighted integral of a concerned response function to the total length as well as the static limits of the equivalent parameters of the material. The static limits could be well predicted with a priori estimation on the structural filling ratio. This analysis makes it possible to evaluate the inherent constraints on the required total length of a passive acoustic treatment before any specific design.

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References

[1] R de L Kronig. On the theory of dispersion of x-rays. *Josa*, 12(6):547–557, 1926.

[2] H A Kramers. La diffusion de la lumiere par les atomes. In *Atti Cong. Intern. Fisica (Transactions of Volta Centenary Congress) Como*, volume 2, pages 545–557, 1927.

[3] K E Peiponen and J J Saarinen. Generalized Kramers–Kronig relations in nonlinear optical-and THz-spectroscopy. *Reports on Progress in Physics*, 72(5):056401, 2009.

[4] J Bechhoefer. Kramers–Kronig, Bode, and the meaning of zero. *American Journal of Physics*, 79(10):1053–1059, 2011.

[5] H W Bode. *Network analysis and feedback amplifier design*. van Nostrand, 1945.

[6] R M Fano. Theoretical limitations on the broadband matching of arbitrary impedances. *Journal of the Franklin Institute*, 249(1):57–83, 1950.

[7] K N Rozanov. Ultimate thickness to bandwidth ratio of radar absorbers. *IEEE Transactions on Antennas and Propagation*, 48(8):1230–1234, 2000.

[8] M Yang, S Chen, C Fu, and P Sheng. Optimal sound-absorbing structures. *Materials Horizons*, 4(4):673–680, 2017.

[9] M M Seron, J H Braslavsky, and G C Goodwin. *Fundamental limitations in filtering and control*. Springer Science & Business Media, 2012.

[10] V Pascalutsa. *Causality Rules: A light treatise on dispersion relations and sum rules*. IOP Concise Physics. Morgan & Claypool Publishers, 7 2018.

[11] F W King. *Hilbert Transforms*, volume 2. Cambridge University Press, 2009.

[12] A Bernland, A Luger, and M Gustafsson. Sum rules and constraints on passive systems. *Journal of Physics A: Mathematical and Theoretical*, 44(14):145205, 2011.

[13] H M Nussenzveig. *Causality and dispersion relations*. Academic Press, 1972.

[14] A Bernland. *Integral Identities for Passive Systems and Spherical Waves in Scattering and Antenna Problems*. PhD thesis, Department of Electrical and Information Technology, Lund University, 2012.

[15] M Gustafsson. An overview of sum rules and physical limitations for passive metamaterial structures. In *META13, the 4th International Conference on Metamaterials, Photonic Crystals and Plasmonics*, 2013.

[16] M Nedic, C Ehrenborg, Y Ivanenko, A Ludvig-Osipov, S Nordebo, A Luger, L Jonsson, D Sjöberg, and M Gustafsson. *Herglotz functions and applications in electromagnetics*, pages 491–514. 12 2020.

[17] M Yang and P Sheng. Sound absorption structures: From porous media to acoustic metamaterials. *Annual Review of Materials Research*, 47:83–114, 2017.

[18] N Jiménez, V Romero-García, V Pagneux, and J-P Groby. Rainbow-trapping absorbers: Broadband, perfect and asymmetric sound absorption by subwavelength panels for transmission problems. *Scientific reports*, 7(1):1–12, 2017.

[19] V Romero-García, N Jiménez, G Theocharis, V Achilleos, A Merkel, O Richoux, V Tournat, J-P Groby, and V Pagneux. Design of acoustic metamaterials made of helmholtz resonators for perfect absorption by using the complex frequency plane. *Comptes Rendus. Physique*, 21(7-8):713–749, 2020.

[20] R Ghaffarivardavagh, J Nikolajczyk, S Anderson, and X Zhang. Ultra-open acoustic metamaterial silencer based on fano-like interference. *Physical Review B*, 99(2):024302, 2019.

[21] S-H Seo, Y-H Kim, and K-J Kim. Design of silencer using resonator arrays with high sound pressure and grazing flow. *Applied Acoustics*, 138:188–198, 2018.

[22] N Jiménez, T J Cox, V Romero-García, and J-P Groby. Metadiffusers: Deep-subwavelength sound diffusers. *Scientific reports*, 7(1):1–12, 2017.

[23] R V Craster and S Guenneau. *Acoustic metamaterials: Negative refraction, imaging, lensing and cloaking*, volume 166. Springer Science & Business Media, 2012.

[24] P A Deymier. *Acoustic metamaterials and phononic crystals*, volume 173. Springer Science & Business Media, 2013.

[25] W S Gan. *New acoustics based on metamaterials*. Springer, 2017.

[26] J Allard and N Atalla. *Propagation of sound in porous media: modelling sound absorbing materials*. 2nd edition, 2009.

[27] G F Carrier, M Krook, and C E Pearson. *Functions of a complex variable: theory and technique*. SIAM, 2005.