A MODAL SUBSTRUCTURING METHOD FOR THE PREDICTION OF THE ACOUSTIC PERFORMANCE IN MUFFLERS

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ABSTRACT

A modal substructuring method is developed for the analysis of the acoustic behaviour of mufflers with uniform cross-section and a perforated pipe carrying mean flow. First, the transversal modes are found by means of a modal synthesis approach based on two different modal bases, which can be obtained analytically for simple geometries, such as circular and elliptical cross-sections, as well as numerically for irregular shapes. Then, the mode-matching technique is applied at the area discontinuities of the muffler in order to obtain its acoustic behaviour. For illustration purposes, a circular concentric resonator with mean flow is analysed. The comparison with finite element results shows good agreement.

1. INTRODUCTION

The acoustic behaviour of mufflers with perforated pipes and mean flow can be analysed by means of different techniques. The use of plane wave models [1] [2] enables to reduce the computational requirements at the cost of a loss of accuracy in the high frequency even below the onset of higher order modes. To improve the prediction, it is possible to use three-dimensional approaches based on analytical solutions of the wave equation [3] for simple geometries, as well as numerical techniques such as FEM [4] or BEM [5] for arbitrary muffler geometry, or a combination of them. This work proposes a technique based on modal synthesis applied to mufflers with uniform cross-section with a perforated pipe carrying mean flow. First, two transversal modal bases (zero pressure and zero velocity in the perforated wall) are considered for the central pipe and the outer chamber, which can obtained analytically or numerically depending on the complexity of the cross-section. The modes are coupled by means of pressure-displacement relationship associated with the perforated pipe. A generalised eigenvalue problem is obtained from which the axial wavenumbers and the transversal modes for the whole cross-section can be evaluated. The use of initially known modal bases leads to a procedure that avoids iteration associated with eigenequations based on Bessel functions [3] [6]. Finally, the mode-matching method is applied at the area discontinuities to calculate the pressure and velocity fields inside the muffler. For illustration purposes, the proposed technique is applied to a concentric resonator, and the results are compared with finite element calculations, showing an excellent agreement.

2. THEORY

Figure 1 shows the geometry of a muffler of arbitrary uniform cross-section with a perforated pipe carrying a mean flow. Here, it is assumed that the mean flow in the outer chamber is negligible.

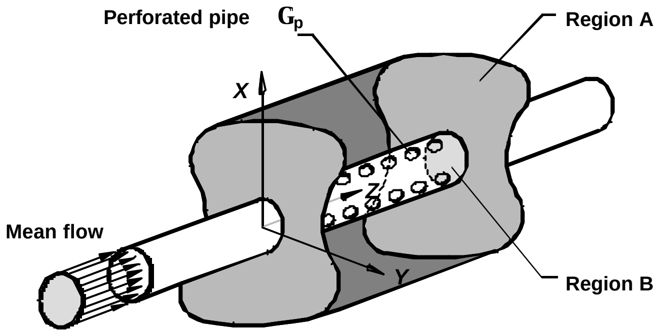

Figure 1. Muffler with perforated pipe carrying a mean flow.

For harmonic behaviour, the acoustic pressure and axial velocity fields are expressed as $P(x,y,z,t)=p(x,y,z)\,e^{j\omega t}$ and $U(x,y,z,t)=u(x,y,z)\,e^{j\omega t}$, respectively, and the acoustic wave equation inside the perforated duct (region B) is given by [1]

$$k^2 p - 2\,jMk\frac{\partial p}{\partial z} + (1-M^2)\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \tag{1}$$

being j the imaginary unit, M the Mach number of the mean flow in the z direction and k the wavenumber ($k = \omega/c_0$, being ω the angular frequency and c_0 the speed of sound). The same equation is also valid for the outer chamber (region A), by considering M = 0. The pressure field in each region of the muffler can be assumed as

$$p(x,y,z) = \Psi(x,y)\,e^{-jk_z z} \tag{2}$$

where k_z is the axial wavenumber. The substitution of Eq. (2) in Eq. (1) yields

$$\left(k^2 - 2Mkk_z - k_z^2(1-M^2)\right)\Psi + \nabla^2\Psi = 0 \quad \leftrightarrow \quad k_t^2\,\Psi + \nabla^2\Psi = 0 \tag{3}$$

being k_t the transversal wavenumber (for region A, the Mach number M must be set to zero in Eq. (3)). The solution of Eq. (3) can be expressed in terms of series of travelling modes, which leads to [1] [3]

$$p(x,y,z) = \sum_{s=0}^{\infty}\left(B_s^+\,e^{-jk_{z,s}^+ z}\,\Psi_s^+(x,y) + B_s^-\,e^{-jk_{z,s}^- z}\,\Psi_s^-(x,y) \right) \tag{4}$$

$$k_{z,s}^{\pm} = \frac{-Mk \mp \sqrt{k^2 - k_{t,s}^2(1-M^2)}}{1-M^2} \tag{5}$$

where $k_{z,s}^{\pm}$ is the axial wavenumber associated with the incident and reflected wave whose modes are $\Psi_s^+(x,y)$ and $\Psi_s^-(x,y)$, respectively, and B_s^+ and B_s^- are propagation coefficients.

The following subsection describes the procedure to evaluate the modal solution in the whole cross-sectional area, which is based on a modal representation of each subdomain (regions A and B).

2.1 Evaluation of the transversal modes

The transversal modes can be obtained from the analysis of the two regions A and B, which have a common perforated boundary Γ_p with an associated acoustic impedance Z_p. The modal solution of each subdomain depends on the boundary conditions given by the perforated pipe over Γ_p. The impedance relates the pressure difference in both sides of the perforated pipe with the particle displacement or velocity [1], and therefore the acoustic fields in each region are coupled. However, two independent modal problems can be considered in each subdomain, corresponding to zero pressure and zero velocity conditions on the perforated boundary. These problems have analytical solution for circular and elliptical cross-sections, which are widely use in automotive mufflers. For more complex geometries, a numerical solution is possible. Thus, the transversal pressure field in each region is expressed by means of

$$\psi(x,y)=\sum_{r=0}^{\infty} q_r^V \phi_r^V (x,y)+\sum_{r=1}^{\infty} q_r^p \phi_r^p (x,y) \tag{6}$$

where ϕ_r^P and ϕ_r^V are the modal pressure fields in this region considering zero pressure and zero velocity boundary conditions in the wall of the perforated pipe, respectively, and q_r^P and q_r^V are participation factors. For a practical solution, the series in Eq. (6) are truncated to a finite number of modes m, and then one has, in matrizant notation

$$\psi(x,y)=[\phi(x,y)]\{q\} \tag{7}$$

Once the transversal pressure has been established, weighted residuals and Galerkin approach are applied to the Eq. (3). A normal pressure gradient is considered as boundary condition over Γ_p. It yields

$$\left([K]-k_t^2[M]\right)\{q\}=\{F\} \leftrightarrow \left(\left([K]-k^2[M]\right)+k_z(2Mk[M])+k_z^2\left((1-M^2)[M]\right)\right)\{q\}=\{F\} \tag{8}$$

where k_t has been replaced as a function of k_z. The previous matrices are defined as

$$[K]=\int_\Omega (\nabla[\phi])^T \nabla[\phi]d\Omega;\ [M]=\int_\Omega [\phi]^T[\phi]d\Omega;$$
$$\{F\}=\int_{\Gamma_p}[\phi]^T\frac{\partial p}{\partial n}d\Gamma=\rho_0\,\omega^2\left(1-M\frac{k_z}{k}\right)^2\int_{\Gamma_p}[\phi]^T\xi_n\,d\Gamma \tag{9-11}$$

being Ω the subdomain considered, ξ_n the particle displacement normal to boundary Γ_p and ρ_0 the fluid density. The vector $\{F\}$ is easily derived considering the linearized Euler equation

$$\xi_n=\frac{1}{\rho_0\,\omega^2\left(1-M\frac{k_z}{k}\right)^2}\frac{\partial p}{\partial n} \tag{12}$$

Eq. (8) can be expressed as

$$\left([K1]+k_z[C1]+k_z^2[M1]\right)\{q\}=\{F\} \tag{13}$$

where $[K1]=[K]-k^2[M]$, $[C1]=2Mk[M]$ and $[M1]=(1-M^2)[M]$. The previous procedure can be applied to both regions (with M = 0 for region A), leading to

$$\left(\begin{pmatrix}[K1^A] & [0]\\ [0] & [K1^B]\end{pmatrix}+k_z\begin{pmatrix}[C1^A] & [0]\\ [0] & [C1^B]\end{pmatrix}+k_z^2\begin{pmatrix}[M1^A] & [0]\\ [0] & [M1^B]\end{pmatrix}\right)\begin{Bmatrix}\{q^A\}\\ \{q^B\}\end{Bmatrix}=\begin{Bmatrix}\{F^A\}\\ \{F^B\}\end{Bmatrix} \tag{14}$$

and, in a compact notation,

$$\left(\left[K1^{AB}\right]+k_z\left[C1^{AB}\right]+k_z^2\left[M1^{AB}\right]\right)\{q\}=\{F\} \tag{15}$$

Both subdomains are related by means of the perforate impedance. The relation $\xi_n^B=-\xi_n^A$ is considered over Γ_p, and, in addition, one has $\xi_n^B=A_p\left(p^B-p^A\right)\frac{1}{j\omega}$, being $A_p=Z_p^{-1}$ the admittance of the perforated pipe. The modal description of the pressure fields gives

$$\xi_n^B=A_p\left(p^B-p^A\right)\frac{1}{j\omega}=A_p\frac{1}{j\omega}\left(\left[\phi^B\right]\left\{q^B\right\}-\left[\phi^A\right]\left\{q^A\right\}\right) \tag{16}$$

and therefore the vector $\left\{F^B\right\}$ can be written as

$$\begin{aligned}\left\{F^B\right\}&=\rho_0\,\omega^2\left(1-M\frac{k_z}{k}\right)^2\frac{1}{j\omega}\int_{\Gamma_p}A_p\left[\phi^B\right]^T\left[\phi^B\right]d\Gamma\left\{q^B\right\}-\rho_0\,\omega^2\left(1-M\frac{k_z}{k}\right)^2\frac{1}{j\omega}\int_{\Gamma_p}A_p\left[\phi^B\right]^T\left[\phi^A\right]d\Gamma\left\{q^A\right\}=\\&=\left(\left[K_p^{BB}\right]+k_z\left[K_{p1}^{BB}\right]+k_z^2\left[K_{p2}^{BB}\right]\right)\left\{q^B\right\}+\left(\left[K_p^{BA}\right]+k_z\left[K_{p1}^{BA}\right]+k_z^2\left[K_{p2}^{BA}\right]\right)\left\{q^A\right\}\end{aligned} \tag{17}$$

Following the same procedure, for M = 0, the load vector associated with region A is

$$\left\{F^A\right\}=\left[K_p^{AA}\right]\left\{q^A\right\}+\left[K_p^{AB}\right]\left\{q^B\right\} \tag{18}$$

The consideration of both subdomains yields

$$\begin{gathered}\begin{Bmatrix}\left\{F^A\right\}\\\left\{F^B\right\}\end{Bmatrix}=\begin{pmatrix}\left[K_p^{AA}\right]&\left[K_p^{AB}\right]\\\left[K_p^{BA}\right]&\left[K_p^{BB}\right]\end{pmatrix}\begin{Bmatrix}\left\{q^A\right\}\\\left\{q^B\right\}\end{Bmatrix}+k_z\begin{pmatrix}[0]&[0]\\\left[K_{p1}^{BA}\right]&\left[K_{p1}^{BB}\right]\end{pmatrix}\begin{Bmatrix}\left\{q^A\right\}\\\left\{q^B\right\}\end{Bmatrix}+k_z^2\begin{pmatrix}[0]&[0]\\\left[K_{p2}^{BA}\right]&\left[K_{p2}^{BB}\right]\end{pmatrix}\begin{Bmatrix}\left\{q^A\right\}\\\left\{q^B\right\}\end{Bmatrix}\leftrightarrow\\\{F\}=\left(\left[K_p\right]+k_z\left[K_{p1}\right]+k_z^2\left[K_{p2}\right]\right)\{q\}\end{gathered} \tag{19}$$

Thus, Eq. (15) can be written as follows

$$\begin{gathered}\left(\left[K1^{AB}\right]-\left[K_p\right]+k_z\left[C1^{AB}\right]-k_z\left[K_{p1}\right]+k_z^2\left[M1^{AB}\right]-k_z^2\left[K_{p2}\right]\right)\{q\}=\{0\}\leftrightarrow\\\left(\left[K1_p^{AB}\right]+k_z\left[C1_p^{AB}\right]+k_z^2\left[M1^{AB}\right]\right)\{q\}=\{0\}\end{gathered} \tag{20}$$

where $\left[K1^{AB}\right]$, $\left[C1^{AB}\right]$ and $\left[M1^{AB}\right]$ are real symmetric matrices with constant coefficients, $\left[K_p\right]$ is a complex symmetric matrix with frequency-dependent coefficients and $\left[K_{p1}\right]$ and $\left[K_{p2}\right]$ are complex non-symmetric frequency-dependent matrices. For a given frequency ω, Eq. (20) enables to obtain the eigenvalues $k_{z,s}$ and the associated eigenvectors $\{q\}_s$, which can be divided into incident and reflected terms $k_{z,s}^+$, $k_{z,s}^-$, $\{q\}_s^+$ and $\{q\}_s^-$. The number of eigenvectors that can be obtained is equal to the dimension of the problem, that is, $2\cdot(2m+1)$. The transversal modes are then evaluated from the modes defined initially for each subdomain and the eigenvectors of Eq. (20)

$$\Psi_s^{A\pm}(x,y)=\left[\phi^A(x,y)\right]\left\{q^A\right\}_s^{\pm}\;;\qquad\Psi_s^{B\pm}(x,y)=\left[\phi^B(x,y)\right]\left\{q^B\right\}_s^{\pm} \tag{21), (22}$$

The solution is approximated and the perforated boundary condition is not satisfied exactly, which can be used as an error indicator. The acoustic pressure in each subdomain can be finally obtained by modal superposition of a given number mt of modal terms that ensure an acceptable accomplishment of the perforated conditions,

$$p_A(x,y,z)=\sum_{s=0}^{mt}\left(A_s^+ e^{-jk_{z,s}^+ z}\,\Psi_s^{A+}(x,y)+A_s^- e^{-jk_{z,s}^- z}\,\Psi_s^{A-}(x,y)\right) \tag{23}$$

$$p_B(x,y,z)=\sum_{s=0}^{mt}\left(B_s^+ e^{-jk_{z,s}^+ z}\,\Psi_s^{B+}(x,y)+B_s^- e^{-jk_{z,s}^- z}\,\Psi_s^{B-}(x,y)\right) \tag{24}$$

Finally, the acoustic velocity is evaluated in each region as

$$u_A(x,y,z)=\frac{1}{\rho_0\,\omega}\sum_{s=0}^{mt}\left(k_{z,s}^+ A_s^+ e^{-jk_{z,s}^+ z}\,\Psi_s^{A+}(x,y)+k_{z,s}^- A_s^- e^{-jk_{z,s}^- z}\,\Psi_s^{A-}(x,y)\right) \tag{25}$$

$$u_B(x,y,z)=\frac{1}{\rho_0\,c_0}\sum_{s=0}^{mt}\left(\frac{k_{z,s}^+}{k-Mk_{z,s}^+}B_s^+ e^{-jk_{z,s}^+ z}\,\Psi_s^{B+}(x,y)+\frac{k_{z,s}^-}{k-Mk_{z,s}^-}B_s^- e^{-jk_{z,s}^- z}\,\Psi_s^{B-}(x,y)\right) \tag{26}$$

In order to obtain the pressure and velocity fields, it is necessary to evaluate the propagation coefficients A_s^+, A_s^-, B_s^+ and B_s^-. The procedure is based on the mode-matching technique, and it is briefly described in the next section.

2.2 Mode-Matching at the area discontinuities

For a given muffler, such as this shown in Figure 1, the propagation coefficients of the pressure and velocity fields given by Eqs. (23) and (24), and those associated with the inlet and outlet pipes, can be evaluated by means of the application of the mode-matching method in the area discontinuities (expansion and contraction). The pressure and velocity conditions in the expansion are

$$p_{Inlet}(x,y,0)=p_B(x,y,0) \quad \text{on } S_B; \quad u_{Inlet}(x,y,0)=u_B(x,y,0) \quad \text{on } S_B; \quad u_A(x,y,0)=0 \quad \text{on } S_A \tag{27}$$

and for the contraction

$$p_{Outlet}(x,y,0)=p_B(x,y,L) \quad \text{on } S_B; \quad u_{Outlet}(x,y,0)=u_B(x,y,L) \quad \text{on } S_B; \quad u_A(x,y,L)=0 \quad \text{on } S_A \tag{28}$$

being L the chamber length. The details of the mode-matching procedure can be found, for example, in [7]. Once the propagation coefficients have been evaluated, it is possible to analyse the acoustic performance of the muffler by means of its transmission loss (TL).

3. RESULTS AND DISCUSSION

An axisymmetric muffler (circular concentric resonator) is considered for illustration purposes, and two different chamber lengths L = 0.15 m and L = 0.3 m are analysed. The radii of the inlet and outlet pipes are 0.025 m, and the radius of the chamber is 0.085 m. The perforated pipe impedance is supposed to be given by means of the expression [1]

$$Z_p=\rho_0\,c_0\,\frac{\left(7.337\cdot 10^{-3}\,(1+72.23\,M)+j\,2.2245\cdot 10^{-5}\,(1+51t)(1+204\,d_h)\dfrac{\omega}{2\pi}\right)}{\sigma} \tag{29}$$

Here, t is the thickness of the perforated pipe, d_h the hole diameter and σ the porosity. The values t = 0.001 m, d_h = 0.003 m and σ = 5% are considered to carry out the calculations. In addition, three values of the mean flow velocity are included in the analysis, M = 0.05, 0.1 and 0.15. Figure 2 (a) shows the results for the chamber length L = 0.15 m, which have been evaluated by means of the present approach. Comparison is given with FEM calculations, with a good agreement in general. The TL is strongly affected by the mean flow value, as expected, and it is shown that a lower Mach number leads to a higher attenuation in the low frequency range, while the opposite trend is observed for higher frequencies. The same comments can applied to Figure 2 (b), in which a chamber length L = 0.3 m is considered. Now, the increase in the length leads a higher number of attenuation domes.

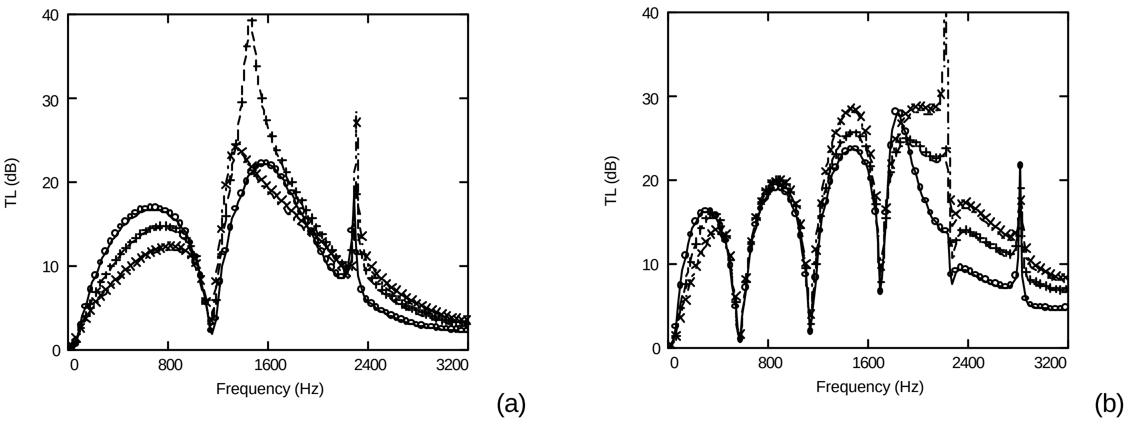

Figure 2. TL of a concentric resonator. (a) L = 0.15 m. (b) L = 0.3 m: ——, M = 0.05, modal method; o, FEM; — —, M = 0.1, modal method; +, FEM; — - —, M = 0.15, modal method; ×, FEM.

4. CONCLUSIONS

A modal substructuring method has been developed for the study of the acoustic performance of mufflers with uniform cross-section and a perforated pipe carrying mean flow. The proposed method predicts the acoustic attenuation without any loss in accuracy in comparison with FEM calculations, and reduces the computational requirements. It can be applied to simple geometries such as circular and elliptical mufflers, for which the pressure modes can be obtained analytically, or complex geometries for which the modes are evaluated numerically.

5. ACKNOWLEDGEMENTS

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