INFLUENCE OF PORE ROUGHNESS ON HIGH-FREQUENCY PERMEABILITY

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ABSTRACT : The high-frequency behavior of the dynamic fluid permeability is studied assuming that the fluid-solid interface appears flat locally. We give a new derivation/expression of the viscous characteristic length Λ. The effect of wedge-shaped intrusions of fixed apex angle is incorporated in an additional higher-order term, non analytic on the viscous skin depth parameter. Precise numerical simulations confirmed the derived dependency on the apex angle.

1. INTRODUCTION

By definition, the dynamic fluid permeability $k(w)$ describes the (linear) response of a simple incompressible fluid entrained in a rigid porous medium and subjected to a harmonic pressure drop across the sample. This response has been the subject of numerous studies and is involved in different problems and applications. As an example, the dynamic permeability is the fundamental ingredient to describe sound propagation in a fluid-saturated rigid-framed porous medium as long as the wavelength is large compared to the characteristic sizes of pores and grains in the medium. The same notion applies when the fluid is a liquid or a gas, and, relaxing the assumption of a rigid frame it may be incorporated in the Biot theory [3] .

Under the assumption that the fluid-solid interface appears flat locally if the viscous skin depth d is small enough, Johnson, Koplik & Dashen [8] have obtained the high-frequency result

$$k(w) = \frac{e^2 f}{a_\infty}(1 - Ce + ...). \qquad (1)$$

Here, $e = \sqrt{n / -iw} = (1+i)d/2$ is the complex viscous skin depth parameter; f, a_∞ and C are purely geometrical parameters, respectively the porosity, tortuosity, and $C = 2/\Lambda$ is such that the "viscous characteristic length" Λ coincide with the pore-size parameter characterizing transport introduced by Johnson, Koplik & Schwartz [9]. We first re-derive the result (1), clarifying one discrepancy with another expression [15,18] of the Λ parameter.

The next term in the bracket is expected to be $O(e^2)$ when the interface has everywhere bounded curvature. Then considering a rugged geometry in the form of 2D corrugated pore channels with wedged-shaped intrusions (see Fig. 1) we derive the high-frequency result

$$k(w) = \frac{e^2 f}{a_\infty}(1 - Ce - C_w e^w + ...), \qquad (2)$$

where the exponent w $(1 < w < 2)$ is related to the apex angle g $(0 < g < p)$ of the wedges :

$$w = \frac{2p}{2p - g}. \qquad (3)$$

We note that the relation (3) is somewhat different from the one proposed in [1], this discrepancy being clarified in the same manner. Precise numerical simulations confirmed the result (3).

2. THE JOHNSON ET AL. HIGH-FREQUENCY DEVELOPMENT (1)

<u>2.1 Definitions and general relations</u>. We define the scaled velocity field $\tilde{\mathbf{v}}$, which solves the following oscillating Stokes flow problem :

$$e^{-2}\tilde{\mathbf{v}} = -\nabla\tilde{p} + \Delta\tilde{\mathbf{v}} + \mathbf{e}\,, \qquad \nabla\cdot\tilde{\mathbf{v}} = 0\,, \qquad \text{in } V_p\,, \tag{4a,b}$$

$$\tilde{\mathbf{v}} = \mathbf{0}\,, \qquad \text{on } S_p\,, \tag{4c}$$

$$\tilde{p} \text{ stationary,} \tag{4d}$$

where $\mathbf{e}$ is the unit macroscopic pressure gradient. (V_p is the pore volume and S_p the pore surface). The dynamic permeability is, by definition, the direct pore volume average [15,18]

$$k(w) = f\frac{1}{V_p}\int_{V_p}\tilde{\mathbf{v}}\cdot\mathbf{e}\,dV\,. \tag{5}$$

Similarly, we define the scaled "electric" field $\mathbf{E}$, which solves the electrical conduction problem

$$\mathbf{E} = -\nabla\Phi + \mathbf{e}\,, \qquad \nabla\cdot\mathbf{E} = 0\,, \qquad \text{in } V_p\,, \tag{6a,b}$$

$$\mathbf{E}\cdot\mathbf{n} = 0\,, \qquad \text{on } S_p\,, \tag{6c}$$

$$\Phi \text{ stationary,} \tag{6d}$$

where $\mathbf{n}$ is the unit outward normal from the pore region. By definition, $\mathbf{E}$ is the microscopic electric field induced in the pore space when a unit macroscopic electric field $\mathbf{e}$ is applied, assuming insulating solid phase and uniform conductivity in the pore space. Its pore volume average is directly related to the tortuosity a_∞ [8, 2] :

$$\frac{1}{a_\infty} = \frac{1}{V_p}\int_{V_p}\mathbf{E}\cdot\mathbf{e}\,dV\,. \tag{7}$$

We note that the word "stationary" [14] means that the fields keep constant pore averaged value (on the average they do not increase or decrease in the direction of $\mathbf{e}$). Using integration by parts it is easily verified that, if j is a stationary field there is the orthogonality relation

$$\int_{V_p}\mathbf{w}\cdot\nabla j\;dV = 0\,, \tag{8}$$

for any divergence-free field $\mathbf{w}$ having zero normal component on the pore surface. Thus, the dynamic permeability and tortuosity factor may be written in equivalent form

$$k(w) = f\frac{1}{V_p}\int_{V_p}\tilde{\mathbf{v}}\cdot\mathbf{E}\,dV\,, \tag{9}$$

$$\frac{1}{a_\infty} = \frac{1}{V_p}\int_{V_p}\mathbf{E}^2\,dV\,. \tag{10}$$

<u>2.2 High-frequency velocity pattern</u>. We now consider the high-frequency limit $e/L \to 0$ of the scaled field $\tilde{\mathbf{v}}$, where L is a characteristic pore size. As argued by Johnson et al. [8], except for a boundary layer of thikness d near the pore walls, the fluid motion is given by potential flow. To leading order we have, $e^{-2}\tilde{\mathbf{v}} \to \mathbf{E}$ and $\tilde{p} \to \Phi$, in the bulk potential flow region. A more exact replacement will be $e^{-2}\tilde{\mathbf{v}} \to \mathbf{E} - \nabla\Pi$ and $\tilde{p} \to \Phi + \Pi$, with Π a small $O(e)$, stationary perturbation induced by the presence of the boundary layer. Assuming that the bounding surface of the pores appears flat locally if the viscous skin depth d is small enough, the perturbation term may be determined by introducing in the analogous electrical conduction problem a layer of variable, vanishing conductivity near the pore walls. The variable conductivity is chosen so as to generate for the current the known variations of the tangential components of the velocity field in the boundary layer. Then, the divergence-free nature of the current necessarily implies, when the interface has a non trivial shape, the existence of normal components near the pore walls that will act as a source for the perturbed potential current in the bulk. Owing to the assumption of locally plane pore walls, the tangential components of the velocity in the boundary layer may be written, to leading order [10]

$$e^{-2}\tilde{\mathbf{v}} = (1 - e^{-b/e})\mathbf{E} \, , \tag{11}$$

where b is a local coordinate measured from the pore walls into the bulk of the pore. We thus consider the perturbed, electrical conduction problem

$$e^{-2}\tilde{\mathbf{v}} = s\,(\mathbf{r})(\mathbf{E} - \nabla\Pi) \, , \qquad \text{in } V_p \, , \tag{12a}$$

$$\nabla \cdot \tilde{\mathbf{v}} = 0 \, , \qquad \text{in } V_p \, , \tag{12b}$$

$$\Pi \text{ stationary,} \tag{12c}$$

$$s\,(\mathbf{r}) = 1 - e^{-b/e} \, . \tag{12d}$$

The field $e^{-2}\tilde{\mathbf{v}}$ is the current induced when a unit electric field is applied, for a medium having insulating solid phase and conductivity $s\,(\mathbf{r})$ in the pore region. Current conservation gives :

$$\nabla \cdot (s\nabla\Pi) = \mathbf{E} \cdot \nabla s \, . \tag{13}$$

In the limit $e/L \to 0$, only derivatives normal to the pore walls need to be considered. Straightforward integration yields the following velocity pattern in the boundary layer :

$$e^{-2}\tilde{\mathbf{v}} = \left(1 - e^{-b/e}\right)\mathbf{E} + e\left(1 - \left[1 + \frac{b}{e}\right]e^{-b/e}\right)\left(\partial_b E_b\right)_{b=0}\mathbf{n} \, . \tag{14}$$

Note that at the same level of approximation there should be present in (14) higher order tangential components that are not obtained in the present reasoning. Setting $b/d = \infty$ in (14) and (12a) we derive the boundary condition, $\partial_b \Pi = e\left(\partial_b E_b\right)_{b=0}$, which applies on the bounding surface of the bulk potential flow region. Obviously a negligible error is introduced by extending this bounding surface to be that of the actual pore walls. The velocity field hence determined in the bulk is

$$e^{-2}\tilde{\mathbf{v}} = \mathbf{E} + e\,\mathbf{N} \, , \tag{15}$$

where the perturbed field $\mathbf{N}$ is the purely geometrical vector field which solves :

$$\nabla \cdot \mathbf{N} = 0 \, , \qquad \text{in } V_p \, , \tag{16a}$$

$$\mathbf{N} \cdot \mathbf{n} = \left(\partial_b E_b\right)_{b=0} \, , \qquad \text{on } S_p \, , \tag{16b}$$

$$\mathbf{N} = \nabla(\text{stationary field}) \, . \tag{16c}$$

<u>2.3 High-frequency permeability</u>. Now evaluating the integral (5) the first term $e^2 f / a_\infty$ in (1) comes, on using (7), from the leading term $\mathbf{E}$ in (15) and the constant tangential term $\mathbf{E}$ in (14). The second term $\frac{e^2 f}{a_\infty} Ce$ splits in two contributions, leading to the new result :

$$\frac{2}{\Lambda} = \frac{\int_{S_p} \mathbf{E} \cdot \mathbf{e} dS}{\int_{V_p} \mathbf{E}^2 dV} + \frac{-\int_{S_p} \Phi \partial_b E_b dS}{\int_{V_p} \mathbf{E}^2 dV} \, . \tag{17}$$

The first is a boundary layer contribution related to the tangential components $-e^{-b/e}\mathbf{E}$ in (14). The second comes from the perturbed bulk potential flow $e\mathbf{N}$ in (15), because $\int_{V_p} \mathbf{N} \cdot \mathbf{e}\, dV = \int_{V_p} \mathbf{N} \cdot \nabla\Phi\, dV$ using the orthogonality (8) between $\mathbf{w} = \mathbf{E}$ and $\nabla j = \mathbf{N}$, and the last integral reduces, after integrating by parts, to the boundary integral $-\int_{S_p} \Phi \partial_b E_b\, dS$. Sheng & Zhou [15,18] erroneously identified $2/\Lambda$ to be the first term in the r.h.s. (17) because they used the incomplete replacement $e^{-2}\tilde{\mathbf{v}} \to \mathbf{E}$. We now show that there is the additional identity $\int_{S_p} \mathbf{E} \cdot \nabla\Phi\, dS = \int_{S_p} \Phi \partial_b E_b\, dS$. Let x^1, x^2, be any Gauss coordinates on the curved pore surface S_p, and choose $x^3 = b$. Given the transformation $x^m \leftrightarrow x^m$ where the x^m are Euclidean coordinates ($x^1 = x$, $x^2 = y$, $x^3 = z$) any tensor known in the Euclidean coordinates may be expressed in the x^m system. (For the reader convenience the present notations allow direct comparisons to be made with the book by Weinberg [17] (Chap. 4)). In this manner, we write $\int_{S_p} \mathbf{E} \cdot \nabla\Phi \, dS = \int_{S_p} E^m \Phi_{;m} dS = \int_{S_p} (E^m \Phi)_{;m} dS - \int_{S_p} \Phi E^m{}_{;m} dS$. The second integral immediately vanishes because $E^m{}_{;m}$ is the divergence of the electric field, which is identically zero. The first integral splits in the first two terms $m = 1,2$, and the third term $m = 3$. Owing to (6c), the latter is nothing but $\int_{S_p} \Phi \partial_b E_b \, dS$; the former vanishes due to the stationary character of Φ. Thus, the identity is proved and adding both terms in (17) we finally obtain, on using (6a) :

$$\frac{2}{\Lambda} = \frac{\int_{S_p} \mathbf{E}^2 dS}{\int_{V_p} \mathbf{E}^2 dV}. \tag{18}$$

A more compact derivation is obtained by evaluating the integral (9). The first term $e^2 f / a_\infty$ is obtained as before, on using (10). The second term then comes uniquely from the boundary layer integral of the field $-e^{-b/e}\mathbf{E}^2$, and the Λ parameter is directly obtained in the form (18). No bulk contribution arises, because of the orthogonality (8) between $\mathbf{E}$ and $\mathbf{N}$.

Eq. (18) was obtained by Johnson, Koplik & Dashen [8] using a classical energetic argument allowing the use of the simple replacements $e^{-2}\tilde{\mathbf{v}} = (1 - e^{-b/e})\mathbf{E}$ in the boundary layer, and $e^{-2}\tilde{\mathbf{v}} = \mathbf{E}$ in the bulk. The role played by the orthogonality relation between the “ground state” field $\mathbf{E}$ and the perturbed field was apparent in [9].
Avellaneda & Torquato [2] (Appendix D) tried to clarify the reason for the discrepancy with Ref. [18]. The role played by the existence of normal components of the velocity near the pore walls, which extend deeply in the fluid, was acknowledged but apparently misinterpreted as a mere “boundary layer” perturbation. The expression (18) was recovered because the representation (9) was finally used.

Note that in straight pore channels ($\mathbf{E} = \mathbf{e}$) the second contribution in (17) vanishes while the first reduces to the pore surface-to-volume ratio S_p / V_p. In general, both contributions are of the same order of magnitude. As an example, they are both equal to S_p / V_p for normal flow through an array of parallel solid cylinders in the dilute limit.

Obviously, when the interface has everywhere bounded curvature the high-frequency development of permeability involves only integer powers of e. It was suggested by Achdou & Avellaneda [1] that the presence of wedged-shaped intrusions would produce higher order terms as indicated in (2), with non integer powers of e related to the apex angle of the wedges. Using the representation (9) of the dynamic permeability we now give a simple argument to determine the relation between the apex angle g and the power of the first higher order term.

3. CORRUGATED PORE CHANNELS

As argued by Achdou & Avellaneda a 2D reasoning is sufficient to study the singularity. The 2D periodic geometry considered is depicted in Fig. 1. The wedge is defined by its external angle a or complementary apex angle $g = 2p - a$. Introducing polar coordinates r, q, we set the origin $r = 0$ on the tip of the wedge and count the angle q from one side of the wedge. The singular potential field $\mathbf{E}(r,q)$ may be written (Landau & Lifshitz [10])

$$E_r = nAr^{n-1}\cos(nq)\,, \quad E_q = -nAr^{n-1}\sin(nq)\,, \tag{19}$$

where n, $1/2 < n < 1$, is the ratio $p/a = p/(2p - g)$ and A is an amplitude factor.

The contribution of the wedges to the integral (9) may be evaluated noting that the velocity field $\tilde{\mathbf{v}}$ matches the value $e^2\mathbf{E}$, to leading order, on the bounding surface of the potential flow region. The separation between this surface and the tip of the wedge shrinks like the viscous skin depth d when frequency increases. Thus, owing to (19), the external potential fields $\mathbf{E}$ and $\tilde{\mathbf{v}}$ must be considered to vary like e^{n-1} and e^{n+1}, respectively, when integrating in the "non plane" boundary layer around the tip of a wedge. Simultaneously, the spatial extent of this modified boundary layer region around the tip shrinks like d^2. It follows that the wedge contribution to (9) will be $O(e^2 e^{n+1} e^{n-1}) = O(e^{2+2n})$, which yields the result (2-3).

Achdou & Avellaneda [1] used the representation (5), while considering only the contribution from the boundary layer, probably due to the aforementioned misinterpretation in [2]. The latter contribution is $O(e^2 e^{n+1}) = O(e^{3+n})$, and they obtain the relation $w = 1 + \dfrac{p}{2p - g}$ between the exponent $w = 2n$ in (2) and the apex angle g. Using the representation (5) the perturbed bulk potential flow induced by the presence of wedges should also be taken into account and it happens that it is now a dominant contribution.

4. NUMERICAL COMPUTATIONS

Numerical computations of the fields $\tilde{\mathbf{v}}$ and $\mathbf{E}$ were performed on the periodic geometry depicted in Fig.1, for different values of the wedge apex angle g and height h. The Stokes problem was solved using the variational formulation of the problem and a N_1 Finite-Element code based on a Uzawa decomposition method. To ensure accuracy, we have used an iterative automatic method, i.e. the solution is computed on the N_1 mesh, next an a-posteriori estimate of the error is computed, and finally the mesh is locally refined accordingly by means of a Delaunay technique developed by Rebay [13]. Successful use of this refinement method on sharp-edges wedges was reported by Firdaouss et al. [7]. Once the flow field $\tilde{\mathbf{v}}$ is known, the dynamic permeability is computed using (5). Coherent calculations of the electric field $\mathbf{E}$ and electrical parameters (7), (18), were obtained using either the Schwartz-Christoffel transformation technique - Cortis & Smeulders [5] - or the method [7].

From (2) it follows that the real part of the dynamic permeability should verify, in a high frequency limit, $Re[k(w)]/d^3 = A + B_w d^{\,w-1}$, where the constant A is related to the formation factor a_∞/f and inverse length C, and the constant B_w is related to w, C_w, and the formation factor. The values of A, B_w, and w can be estimated by comparison between the high-frequency numerical data for $Re[k(w)]/d^3$ and the above theoretical form. As an example, we show below the results obtained for the exponent w when the wedge angle g varies between 0 and $p/2$. The wedge height h is set 0.5.

In the singular limit of "knife-shaped" intrusions ($g = 0$), the value $w = 1$ indicates the merging of the different terms. For flat surfaces $g = p$, the value $w = 2$ will be obtained. The computed data is relatively close to the theory, the Achdou & Avellaneda prediction being plotted for comparison. As compared to the situation encountered in smooth pore channels, the effect of sharp wedges is to produce a much slower convergence of the high-frequency dynamic permeability to the Johnson et al. development (1). In these situations, the development (2) enables representing in a very accurate manner the high-frequency permeability [6].

5. CONCLUSION

Analyzing in detail the fluid velocity pattern established in non trivial geometry in the high-frequency limit, we have provided a new derivation of the Johnson et al. [8] high-frequency development of the dynamic permeability and a new expression of the characteristic length Λ. Two different contributions to the dynamic permeability are now apparent. One comes from the boundary layer near the pore walls, another comes from a perturbed potential flow in the bulk, which is induced in non trivial geometry by the presence of the boundary layer. This understanding has been applied to derive the correct form of the leading higher order terms which are present in rugged geometry. Such terms are essential to obtain the correct high-frequency behavior of the dynamic permeability when sharp edges are present.

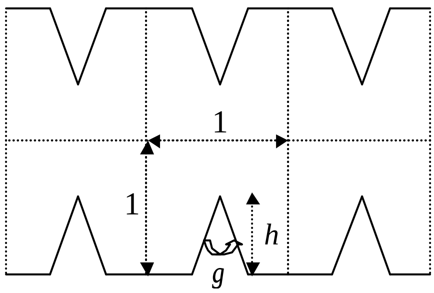

FIG. 1. Geometry of the 2D rugged pore channel

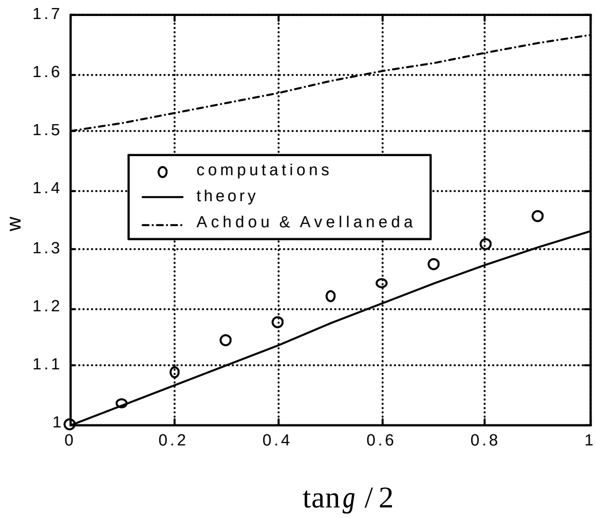

FIG. 2. Dependence of the exponent w on the wedge apex angle g .

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