AN APPROACH OF THE CALCULATION OF THE SPECTRUM OF CONICAL REED INSTRUMENTS BY USING THE VARIABLE TRUNCATION METHOD

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ABSTRACT The variable truncation method has been used first for cylindrical reed instruments. Thanks to the separation of odd and even harmonics, it is possible to justify it rigorously by comparing it to the exact solution obtained when losses are ignored : the Helmholtz motion. A generalization to a particular case of conical instruments, equivalent to stringed instruments bowed at the third of the string length, is treated. The harmonics are separated into three classes ; the first step of the variable truncation considers only two harmonics, and leads to interesting results concerning the different thresholds and the spectrum for large oscillations.

INTRODUCTION

The harmonic balance technique is used since about 20 years for the computation of the spectrum, during the steady-state regime, of self-sustained oscillations of musical instruments. Recently, an approximate version of the method, called the variable truncation method (VTM), has been used in order to obtain simple formulae for clarinet-like instruments [1,2,3]. The justification of the VTM was based on the comparison with the solution for lossless resonators, i.e. square signals corresponding to the case of the well known Helmholtz motion, and it was proved that the method was an extension of the "small oscillations" method, valid only near the oscillations thresholds [4]. The present paper examines a generalization of the VTM, for the case of very simplified conical reed instruments. A particular interest lies in the existence of inverse bifurcations from the static, non oscillating regime : for this case the "small oscillations" method is of less interest because certain stable oscillating regimes are far from the threshold of instability of the static regime.

The VTM can be a priori used for any shapes of resonators, but the paper is limited to resonators for which the lossless limit is a Helmholtz motion, in order to know a limit solution, and thus the validity of the results. It is the reason why we start with a resonator equivalent to a cylindrical tube excited by a reed at the third of its length, further works being possible for other ratios, like 1/4, 1/5,etc... In recent papers, these kinds of resonators have been proved to be equivalent to "stepped cones", conical resonators being a limit case of this class of shapes [3]. Therefore the present study can be regarded as a first step to the study of saxophones.

The paper presents first the model of the resonator and the excitation system, and the Helmholtz motion solution. Then approximate solutions are obtained when losses are present and the different oscillating regimes (standard, octave, and inverse Helmholtz motion) are discussed, especially oscillation thresholds and spectra. Further works are finally presented.

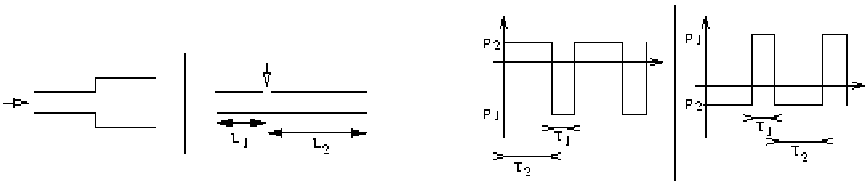

Figure 1 : a stepped cone and its equivalent (the arrows indicate the location of the mouthpiece) ; standard and inverse Helmholtz motion.

THE MODEL; HELMHOLTZ MOTION FOR A LOSSLESS RESONATOR (see ref. [3])

For the reed and mouthpiece, the classical model based on the Bernoulli equation (with some assumptions) and the description of the reed as a simple spring leads to the following equation relating acoustic volume velocity u(t) and pressure p(t) at the entrance of the resonator [3] :

$$u^2 = u_{00}^2 + Ap + Bp^2 + Cp^3 \quad . (1)$$

The values of the coefficients are the following :

$$u_{00}^2 = \zeta^2(1-\gamma)^2\gamma \;\; ; \;\; A = \zeta^2(1-\gamma)(3\gamma-1) \;\; ; B = \zeta^2(3\gamma-2) \; ; \; C = -\zeta^2$$

where γ is the (excitation) pressure in the mouth, assumed to be constant, and ζ is the "embouchure" parameter, characterizing the reed and the reed aperture. These two parameters and the two acoustic quantities are dimensionless. The coefficients are time-independent, and only the coefficient A varies significantly with the mouth pressure γ in the range of practical interest : the reed being assumed not to beat, γ is limited to 0.5. Thus B and C are negative and almost constant, and A changes of sign for $\gamma = 1/3$, i.e. the oscillation threshold for a lossless cylindrical (clarinet-like) resonator. A is opposite to a resistance : if it is positive, the resistance is negative, and oscillation is possible.

Considering now the resonator, the mouthpiece excites a cylindrical tube at the third of its length. If no losses are taken into account, a pure Helmholtz motion is obtained. The fundamental regime has its frequency as $f_1 = c/2\ell$, where ℓ is the length of the resonator, and c is the speed of sound. Two episodes of the period T need to be distinguished : the shorter, of duration $T_1 = T/3$, with a pressure value of p_1 , and the longer, of duration $T_2 = 2T/3$, with a pressure value of p_2. If p_2 is positive, the motion is called "standard Helmholtz", if it is negative, the motion is called "inverse Helmholtz" (see figure 1).

When no losses are taken into account, the impedance is zero for the frequencies $3mf_1$, where m is an integer, and the corresponding component of the pressure vanish. It can be also proved that the volume velocity needs to be constant and if the impedance at zero frequency is zero, the mean pressure is also zero, thus :

$$u(p_1) = u(p_2) \quad (2) \quad \text{and} \quad T_1\, p_1 + T_2\, p_2 = 0 \quad \text{or} \quad p_1 = -2p_2 \, . \; (3)$$

From equations (1), (2) and (3) the following values are deduced for the pressure :

$$p_2 = 0 \quad \text{or} \quad p_2 = \frac{B}{6C}\left[1 \pm \sqrt{1+\alpha}\right] \quad (4)$$

where $\alpha = -AC/B^2$. The zero value corresponds to the static regime, and the non zero values to the oscillating regimes. $\alpha = 0$ can be proved to be the threshold of instability of the static regime. For the further analysis, it is useful to expand the formula (4) for small values of α :

$$p_2\frac{C}{B} = \frac{1}{3}(1+3\alpha-9\alpha^2+O(\alpha^3)) \quad (4a) \quad \text{or} \quad p_2\frac{C}{B} = -\alpha(1-3\alpha+O(\alpha^2)) \quad . \; (4b)$$

Figure 2 shows the two oscillating solutions with respect to α , i.e. the mouth pressure.

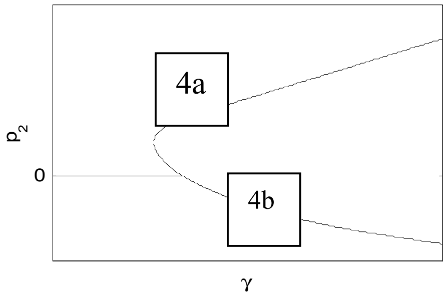

Figure 2 : solutions with respect to the mouth pressure (equation 4)

DECOMPOSITION OF THE SIGNAL

In order to simplify the analysis when losses are present, it is possible to decompose the spectrum of the signal into three kinds of harmonics, the number of which being equal to n=3q, n=3q+1, n=3q-1, respectively where q is an integer, as follows :

$$p(t)=p_s(t)+p_a(t)+p_a^*(t) \quad (5)$$

where, for a square signal, $p_s(t)=0$. Because the acoustic pressure signal is real, the last term is the complex conjugate of the second one. For a square signal corresponding to a direct Helmholtz motion, if the first harmonic is chosen to have a zero phase, the second term is found to be :

$$p_a(t)=\frac{3\sqrt{3}}{2\pi}p_2\sum_{q=-\infty}^{+\infty}\frac{1}{3q+1}e^{j(3q+1)\omega t} \quad (6)$$

where ω is the angular frequency. This expression leads to : $p_a(t)=0.5(1+j\sqrt{3})p_2$ for 0 < t < 2T$\pi/3$; $p_a=-p_2$ for 2T$\pi/3$ < t < 4T$\pi/3$; $p_a(t)=0.5(1-j\sqrt{3})p_2$ for 4T$\pi/3$ < t < T .

DECOMPOSITION OF THE NONLINEAR EQUATION

The direct solving of equation (1) with the input impedance condition for the calculation of the steady-state regime, when losses are taken into account, is rather intricate. It is possible to simplify the problem, by considering that the impedance of the first kind of harmonics is very small.

Using decomposition (5), and a similar one for the volume velocity u(t), equation (1) can be split into two equations. These equations are obtained by using simple rules concerning the products of the different terms in equation (5), and by assuming that the pressure $p_s(t)$ is very small, the corresponding impedance being small. A third equation is obtained, but it is the conjugate of the second one. A treatment of the two equations is possible, as for the case of a clarinet-like instrument. Nevertheless a simplification of equation (1) can be obtained by calculating a series expansion of the volume velocity, as follows :

$$u=u_{00}+A'p+B'p^2+C'p^3 \quad (7)$$

where the new coefficients are easy to determine. When no losses are present, u(t) is constant, and all equations (4), (4a), (4b) remain valid by replacing the coefficients A, B and C by the new coefficients. New equations are obtained, the second one being decoupled from the first one ;

$$u_s=u_{00}+2B'\left|p_a\right|^2+C'(p_a^3+p_a^{3*}) \quad (8) \qquad u_a=A'p_a+B'p_a^{2*}+3C'\left|p_a\right|^2 p_a\,. \quad (9)$$

Equation (11) is the most important, the solving of (10) being deduced in a second step. Several techniques can be used : the complete (numerical) harmonic balance technique, the perturbation of the Helmholtz motion solution, and the VTM, used below.

THE VARIABLE TRUNCATION METHOD (VTM)

The first order of the VTM is the so-called approximation of the first harmonic. In the present case, it is clearly not interesting, because the Helmholtz motion is not symmetrical : at least two harmonics need to be considered. We are seeking for $p_a(t)=P_1\,e^{j\omega t}+P_2^*e^{-2j\omega t}$ (where P_1.is real).

The successive higher order harmonics can be deduced successively. Using equation (9), the following approximate system is obtained:

$$Q_1\,(Q_1^2+2\left|Q_2\right|^2+\tfrac{1}{3}(2Q_2-\alpha_1))=0 \quad (10) \qquad 27\left|Q_2\right|^2Q_2+6(\left|Q_2\right|^2+2Q_2^2)+(2-6\alpha_1+3\alpha_2)Q_2-\alpha_1=0 \quad (11)$$

where $Q_i = P_i C'/B'$ and $\alpha_i = (Y_i - A')C'/B'^2$, Y_i. being the input admittance for the harmonic i. It enlarges a result given by [5] for a system with only two resonance peaks, assumed to have harmonically related frequencies. These equations can be solved by expansion with respect to the parameters α, and the two solutions generalizing (4a) and (4 b) can be found.

APPROXIMATE EXPRESSIONS NEAR THE OSCILLATION THRESHOLDS

At the second order of the quantities α_i, the following result is obtained:

$$Q_1^2 = \frac{\alpha_1 \alpha_2}{2} - \frac{3}{8}\alpha_1 \left[2(2\alpha_1 + \alpha_2)(\alpha_1^* + \alpha_2) - 3\left|\alpha_1\right|^2 \right] \quad (12) \quad \text{and} \quad Q_2 = \frac{\alpha_1}{2}\left[1 - \frac{3}{2}(\alpha_1^* + \alpha_2)\right] . (13)$$

When no losses are present ($y_1 = y_2 = 0$), one get

$$Q_1 = \frac{|\alpha|}{\sqrt{2}}(1 - \frac{27}{8}\alpha) \quad \text{and} \quad Q_2 = \frac{\alpha}{2}(1 - 3\alpha)$$

instead of the exact result, given from equations (4b) and (6) :

$$Q_1 = \frac{3\sqrt{3}|\alpha|}{2\pi}(1 - 3\alpha) \quad \text{and} \quad Q_2 = \frac{3\sqrt{3}\alpha}{4\pi}(1 - 3\alpha) .$$

The VTM produces an error of 17% for the first harmonic (10% for a clarinet-like instrument, see [1]), and 18% for the second one. Another calculation for the other solution, corresponding to (4a) gives the same order of magnitude. It is therefore possible to use the approximation when losses are taken into account, and the solutions are compatible with the results of [4] near the threshold of the unstable motion..

For small but non necessarily zero inharmonicity of the two first resonance frequencies, equations (12) and (13) lead to the following result :

$$Q_1^2 = \frac{1}{2}(\alpha - \mathrm{Re}(y_1))(\alpha - \mathrm{Re}(y_2))$$

A separation of the thresholds of two regimes is obtained : no solutions exist between the values of the real part of the input admittance for the two resonance frequencies ; for $\alpha < \mathrm{Re}(y_1)$, the solution is similar to the solution (4b), for negative α, and the motion is an unstable, standard Helmholtz motion, the second harmonic vanishing at the threshold. For $\alpha > \mathrm{Re}(y_2)$, we get an inverse, stable Helmholtz motion, but its amplitude starts with a non zero value, because the second harmonic does vanish at threshold (see equation (13)). When losses tend to zero, this solution tends to the solution (4b) for positive α, with no gaps between these two solutions. This situation was described in [5, figure 8], from numerical solutions. A third regime exists also : the octave regime, similar to the fundamental regime of a clarinet-like instrument, i.e. with a direct bifurcation.

CONCLUSION

The VTM allows to describe analytically and to understand the separation of thresholds due to losses. It also allows to get solutions at a finite distance of the thresholds. Further work is in progress concerning the higher harmonics, the normal regime (4a), with its subcritical threshold, its spectrum and playing frequency, and the use of another method, more appropriate for large oscillations, based on the perturbation of the Helmholtz motion.

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