

# Subspace projection of Helmholtz problems for the efficient solution over frequency bands

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**ABSTRACT:** A subspace projection methodology providing simultaneous solutions of the Helmholtz equation at multiple frequencies is presented. The subspace projector is obtained with an unsymmetric block Lanczos algorithm applied to a transfer function that is derived from the finite element discretization of the Helmholtz problem. This approach is equivalent to a matrix-valued Padé approximation of the transfer function. The proposed method is applicable to unsymmetric systems and allows the treatment of a much wider range of practical problems, including near-field and fluid-structure interaction computations. Numerical examples illustrate the efficiency of the method.

## INTRODUCTION

In many engineering applications, the solution of the Helmholtz equation over a *frequency window* is of interest. Such applications include acoustic car design as interior problem and acoustic radiation of sound from vibrating structures as exterior example. Commonly, the approach is to repeatedly solve the complete discretized systems of equations for each frequency of interest. As system sizes increase, this becomes a computationally expensive, if not prohibitive, task. Moreover, for a wide range of applications, the solution at each point of the discretized domain may not be of interest. Rather it is desired to obtain the solution at a certain subset of the computational domain. Such *partial fields* include single points in the *near-field* of a vibrating structure or an enclosing surface such as a sphere for *far-field* computations.

To improve the computational efficiency of such *multiple-frequency partial-field* problems outlined above, a *Krylov subspace projection method* is here presented. The method efficiently projects the full matrix system to one of much smaller dimension, allowing by this for the simultaneous solution over a frequency window in one step. The finite element method is used for the discretization of interior or exterior field problems. In the case of an exterior field problem, a Dirichlet-to-Neumann (DtN) map (Keller and Givoli, 1989) is applied on the truncating surface. The matrix system is recast into an *unsymmetric transfer function* form that filters the partial field out of the full solution vector. The transfer function is further reformulated into a *standard shifted form*. This form involves the inverse of the frequency-dependent system matrix, which can not be evaluated directly. Hence, a reduced-dimension system is computed by an oblique projection of the matrix onto a Krylov subspace. The projection is obtained by applying an unsymmetric block Lanczos algorithm of Aliaga et al. (2000) on the standard shifted form.

## MULTIPLE-FREQUENCY ANALYSIS OF THE INTERIOR ACOUSTICS PROBLEM

An inviscid compressible fluid in the domain  $\Omega$  with boundary  $\Gamma$  is treated. Only time-harmonic vibration with angular frequency  $\omega$  is considered. The acoustic wave number  $\kappa$  is defined as  $\kappa = \omega/c$ , where  $c$

is the speed of sound. The variational form of the acoustic problem is discretized with finite elements. Continuous piecewise-linear polynomials are chosen and the same interpolation functions are used for the trial solution and weighting functions. Introducing the  $N \times 1$  dimensional vector  $\mathbf{p}$ , containing unknown nodal values of the pressure field, the system of equations

$$[\mathbf{K} - \kappa^2 \mathbf{M}] \mathbf{p} = \mathbf{f} \quad (1)$$

is obtained (see Hughes, 1987).  $N$  denotes the total number of unknowns *excluding* the nodal components where Dirichlet data is prescribed. Moreover,  $\mathbf{f}$  is the  $(N \times 1)$  - dimensional load vector.  $\mathbf{K} \in \mathbb{R}^{N \times N}$  is a symmetric positive semi-definite matrix and  $\mathbf{M} \in \mathbb{R}^{N \times N}$  is symmetric positive definite. Both matrices are independent of the wave number  $\kappa$ .

### Points In The Computational Domain

As outlined in the introduction, the focus here is on the efficient computation of certain restricted regions in the computational domain. For an interior field problem, this is equivalent to choosing a subset of nodal points from  $\mathbf{p}$ . Arranging  $N_{\text{nf}}$  basis vectors  $\mathbf{e}_j, j = 1, \dots, N_{\text{nf}}$  in a restriction operator  $\mathbf{E} \in \mathbb{R}^{N \times N_{\text{nf}}}$  gives rise to a so-called *transfer function*  $\mathbf{H} \in \mathbb{R}^{N_{\text{nf}} \times N_{\text{nf}}}$

$$\mathbf{H} = \mathbf{E}^T [\mathbf{K} - \kappa_i^2 \mathbf{M}]^{-1} \mathbf{f}, \quad (2)$$

where  $\mathbf{H}$  contains the pressure values of the points in the restricted region of interest. As already pointed out, the results in  $\mathbf{H}$  are of interest at *multiple wave numbers*  $\kappa_i, i = 1, \dots, N_f$ . To obtain Eq. (2) for all these wave numbers, the inversion of Eq. (1) is mandatory for each  $\kappa_i$ . This is computationally expensive. Moreover, the restriction to a small subset of all points is not exploited, since the system matrix must be factorized in each step. To take advantage of the restriction to a subset for the solution over a frequency window, Eq. (2) is reformulated to a standard shifted form allowing the application of a Krylov subspace projection that reduces the system to a much lower dimension. This system is then feasible to be solved over a frequency range with almost no extra computational cost, while preserving the accuracy of the solution.

First, a *frequency-shift* parameter  $\sigma_i$  and a chosen reference value  $\kappa_0$  are introduced,

$$\sigma_i := \kappa_i^2 - \kappa_0^2, \quad i = 1, \dots, N_f. \quad (3)$$

This results in the expression  $\mathbf{H}(\sigma_i) = \mathbf{E}^T [\mathbf{A}_0 - \sigma_i \mathbf{M}]^{-1} \mathbf{f}$  with  $\mathbf{A}_0 = \mathbf{K} - \kappa_0^2 \mathbf{M}$ . If  $\kappa_0$  is not coinciding with an eigenfrequency of the system, the matrix  $\mathbf{A}_0$  is nonsingular and has thus a unique inverse. Denoting  $\mathbf{L} := \mathbf{E} \in \mathbb{R}^{N \times N_{\text{nf}}}$ ,  $\mathbf{R} := \mathbf{A}_0^{-1} \mathbf{f} \in \mathbb{R}^{N \times 1}$ , and  $\mathbf{A} := \mathbf{A}_0^{-1} \mathbf{M} \in \mathbb{R}^{N \times N}$ , the *unsymmetric standard shifted form*

$$\mathbf{H}(\sigma_i) = \mathbf{L}^T [\mathbf{I} - \sigma_i \mathbf{A}]^{-1} \mathbf{R} \quad (4)$$

is obtained. Malhotra and Pinsky (2000) derive a symmetric form. This paper advances this formulation to an *unsymmetric* form by allowing  $\mathbf{L} \neq \mathbf{R}$  and  $\mathbf{A} \neq \mathbf{A}^T$ .

### Krylov Subspace Projection Method

In the following, we briefly describe the block Krylov subspaces used herein and the block Lanczos algorithm used to compute bases for them. The left and right  $n$ -th block Krylov subspaces are defined as

$$\mathcal{K}_n(\mathbf{A}^T, \mathbf{L}) = \text{span}\{\mathbf{L}, \mathbf{A}^T \mathbf{L}, \dots, \mathbf{A}^{T \frac{n}{m}-1} \mathbf{L}\} \quad \text{and} \quad \mathcal{K}_n(\mathbf{A}, \mathbf{R}) = \text{span}\{\mathbf{R}, \mathbf{A} \mathbf{R}, \dots, \mathbf{A}^{\frac{n}{p}-1} \mathbf{R}\}. \quad (5)$$

Note that the number  $n$  denotes the dimension of the block Krylov subspace and that we assume for simplicity that all columns are linear independent. Since in each step blocks of column-size  $m$  and  $p$  are added to the left and right Krylov subspaces, the number of blocks is limited to  $\frac{n}{m}$  and  $\frac{n}{p}$ , respectively. The matrix  $\mathbf{A} \in \mathbb{C}^{N \times N}$  is the system matrix and  $\mathbf{L} \in \mathbb{C}^{N \times m}$  and  $\mathbf{R} \in \mathbb{C}^{N \times p}$  are blocks of left and right starting vectors, respectively. The algorithm of Aliaga et al. (2000) applied in this work mainly

differs from other block versions of the Lanczos algorithm (see Freund, 2000) in that it allows to choose different block sizes for the left and right starting blocks,  $m \neq p$ , which is a necessary feature for our problem, since in general the block sizes of  $\mathbf{L}$  and  $\mathbf{R}$  are different.

The unsymmetric block Lanczos algorithm is based on three-term recurrences that provide a projection of a  $N \times N$ -dimensional matrix  $\mathbf{A}$  onto the Krylov-subspace  $\mathcal{K}_n(\mathbf{A}, \mathbf{R})$  of much lower dimension  $n \times n$ . It creates two bi-orthogonal sequences

$$\mathbf{W}_n = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n] \in \mathbb{C}^{N \times n} \quad \text{and} \quad \mathbf{V}_n = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] \in \mathbb{C}^{N \times n}, \quad (6)$$

which form bases for the left and right Krylov subspaces in Eq. (5). The bi-orthogonality condition is expressed in matrix form as  $\mathbf{W}_n^T \mathbf{V}_n = \mathbf{\Delta}_n = \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$  with  $\delta_i \in \mathbb{C}$  being the bi-orthogonality factors. With this notation, the actual three-term recurrences in the block Lanczos algorithm can be written in matrix form as

$$\mathbf{A}^T \mathbf{W}_n = \mathbf{W}_n \mathbf{\Delta}_n^{-1} \mathbf{T}_n^T \mathbf{\Delta}_n + [\mathbf{0}_{N \times (n-m)}, \mathbf{W}_{(\ell)}] \quad \text{and} \quad \mathbf{A} \mathbf{V}_n = \mathbf{V}_n \mathbf{T}_n + [\mathbf{0}_{N \times (n-p)}, \mathbf{V}_{(r)}]. \quad (7)$$

The rightmost terms  $\mathbf{W}_{(\ell)}$  and  $\mathbf{V}_{(r)}$  constitute the last blocks computed in the iteration. For these equations, the projection matrix  $\mathbf{T}_n$  is defined. It is a banded matrix with lower and upper maximum bandwidth  $m$  and  $p$ , respectively, whose entries are the coefficients computed during the iteration steps of the block Lanczos algorithm. The matrix  $\mathbf{T}_n$  constitutes a projection of  $\mathbf{A}$  onto the right block Krylov subspace  $\mathcal{K}_n(\mathbf{A}, \mathbf{R})$ . The block Lanczos algorithm has to deal with two main difficulties which are breakdowns due to division by zero and the premature termination of the algorithm due to linear dependent vectors in the Krylov subspaces. Both problems are remedied by *look-ahead steps* that avoid the division by zero and by *vector deflation* that eliminates linear dependent basis vectors. To keep the description brief, neither case is assumed to occur here. For details, the reader is referred to Aliaga et al. (2000).

After the projection matrices and subspace bases are available, a reduced-dimension system of Eq. (4) can be obtained by projecting the matrices from the right and left onto the Krylov subspaces  $\mathcal{K}(\mathbf{A}^T, \mathbf{L})$  and  $\mathcal{K}(\mathbf{A}, \mathbf{R})$  defined by the bases  $\mathbf{W}_n$  and  $\mathbf{V}_n$ , respectively. The *reduced-dimension shifted system* is derived as

$$\mathbf{H}_n(\sigma_i) = (\mathbf{\Delta}_n^T \boldsymbol{\eta}_n)^T [\mathbf{I}_n - \sigma_i \mathbf{T}_n]^{-1} \boldsymbol{\rho}_n \quad (8)$$

and is denoted by the index  $n$  as the  $n$ -th order reduced-dimension system. The matrices  $\boldsymbol{\eta}_n$  and  $\boldsymbol{\rho}_n$  contain biorthogonalization factors of the starting blocks  $\mathbf{L}$  and  $\mathbf{R}$ , respectively.

The Krylov subspace projection replaces the  $N$ -dimensional matrix problem in Eq. (4) in  $\mathbf{A}$  by an approximation of dimension  $n$ ,  $n \ll N$ , in  $\mathbf{T}_n$ . This feature will be exploited for our acoustic problem to obtain simultaneous solutions at multiple frequencies.

## DISCRETIZATION OF AN EXTERIOR ACOUSTICS PROBLEM

In this section, the methodology is extended to exterior acoustics. In order to make the boundary-value problem feasible for a solution with the FEM, the infinite-domain problem is transformed into an equivalent statement in a bounded domain. For this, the infinite domain is truncated by a surface  $\Gamma_{\text{DtN}}$  and the condition  $\nabla p \cdot \mathbf{n} = -\mathcal{B}_{\text{DtN}}(p)$  on  $\Gamma_{\text{DtN}}$  is introduced, imposing the DtN map of Keller and Givoli (1989) as an exact non-reflecting boundary condition. Explicit forms of the DtN operator can be obtained for separable surfaces such as a circle and a sphere in the form

$$\mathcal{B}_{\text{DtN}}(p) = \sum_{n=0}^{\infty} z_n(\kappa R) \int_{\Gamma_{\text{DtN}}} s_n(\mathbf{x}, \boldsymbol{\xi}) p(\boldsymbol{\xi}) d\Gamma_{\boldsymbol{\xi}}, \quad \mathbf{x} \in \Gamma_{\text{DtN}}, \quad (9)$$

where the DtN kernels  $s_n(\mathbf{x}, \boldsymbol{\xi})$  are the *surface harmonics* on  $\Gamma_{\text{DtN}}$ , the  $z_n(\kappa R)$  can be identified as *radial-impedance* coefficients and  $R$  denotes the radius of the circle or sphere. In a computation, the series in Eq. (9) will be truncated after  $N_{\text{DtN}}$  terms. It is well known (see Grote and Keller, 1995) that this truncation may lead to a singular system matrix. For reasons that will be apparent in the next section, we modify the DtN condition by adding and subtracting a local  $\mathcal{B}_1(\kappa)$  condition of Bayliss et al. (1982), such that  $\mathcal{B}_{\text{DtN,mod}}(p) = (\mathcal{B}_{\text{DtN}} - \mathcal{B}_1)(p) + \mathcal{B}_1(p)$ . For a detailed description of the

approach see Wagner et al. (2002). The variational form of the exterior acoustic problem is discretized with finite elements, yielding the system of equations

$$\left[ \mathbf{K} - \kappa^2 \mathbf{M} + \mathbf{B}_1(\kappa) + \mathbf{K}_{\text{DtN}}(\kappa) \right] \mathbf{p} = \mathbf{f}, \quad (10)$$

see Wagner et al. (2002) for the DtN matrices  $\mathbf{K}_{\text{DtN}} \in \mathbb{C}^{N_{\text{DtN}} \times N_{\text{DtN}}}$  and  $\mathbf{B}_1 \in \mathbb{C}^{N_{\text{DtN}} \times N_{\text{DtN}}}$ .

It was shown by Malhotra and Pinsky (2000) that  $\text{rank}(\mathbf{K}_{\text{DtN}}) = N_{\text{mod}}$ ,  $N_{\text{mod}} \ll N$ , and

$$\boxed{\mathbf{K}_{\text{DtN}} = \mathbf{F} \mathbf{\Lambda} \mathbf{F}^T}, \quad (11)$$

where  $N_{\text{mod}} = 2N_{\text{DtN}} + 1$  in 2D,  $\mathbf{F} \in \mathbb{R}^{N \times N_{\text{mod}}}$ , and  $\mathbf{\Lambda} \in \mathbb{C}^{N_{\text{mod}} \times N_{\text{mod}}}$ . The non-zero elements in each column of  $\mathbf{F}$  represent the discrete surface harmonics associated with  $\Gamma_{\text{DtN}}$ .  $\mathbf{F}$  has full column rank. It is important to note that  $N_{\text{mod}}$  is generally very small compared to  $N$ ,  $N_{\text{mod}} \ll N$ .  $\mathbf{\Lambda}$  is a diagonal  $N_{\text{mod}} \times N_{\text{mod}}$  dimensional matrix containing the impedance coefficients  $z_n$ . As such, the elements of  $\mathbf{\Lambda}$  depend on the wave number  $\kappa$ , while  $\mathbf{F}$  is frequency independent. A more detailed description of those matrices is given by Malhotra and Pinsky (2000) and Wagner et al. (2002).

## MULTIPLE-FREQUENCY PARTIAL-FIELD SOLUTIONS IN EXTERIOR ACOUSTICS

To compute far-field solutions, we require  $p$  and  $\nabla p \cdot \mathbf{n}$  for  $\mathbf{x} \in \Gamma_{\text{DtN}}$ . By expressing the pressure field on  $\Gamma_{\text{DtN}}$  in a Fourier series, we can reduce the problem of far-field computations to one where only the modal coefficients in such an expansion need to be computed:

$$p(R, \theta) = a_0 + \sum_{n=1}^{N_{\text{DtN}}} (a_n \cos n\theta + b_n \sin n\theta). \quad (12)$$

Observe that this approach yields a representation of  $p$  equivalent to the DtN series expression in Eq. (9). Especially, the number of terms in the series must be chosen equal to the DtN series expansion for reasons apparent in the following section. The resulting modal coefficients can be written as a complex valued  $(N_{\text{mod}} \times 1)$ -dimensional transfer function

$$\mathbf{H}_{\text{mod}} := [a_0, a_1, \dots, a_{N_{\text{DtN}}}, b_1, \dots, b_{N_{\text{DtN}}}]^T = \mathbf{D} \mathbf{F}^T \mathbf{p}, \quad (13)$$

where  $\mathbf{H}_{\text{mod}} \in \mathbb{C}^{N_{\text{mod}} \times 1}$ ,  $N_{\text{mod}} = 2N_{\text{DtN}} + 1$ , the columns of  $\mathbf{F} \in \mathbb{R}^{N \times N_{\text{mod}}}$  are the discrete surface harmonics identical to those in Eq. (11), and  $\mathbf{D}$  is a diagonal  $N_{\text{mod}} \times N_{\text{mod}}$  matrix containing the scalar orthonormalization constants of the discrete transformation.

Defining  $\mathbf{H} \in \mathbb{C}^{N_{\text{pf}} \times 1}$ ,  $N_{\text{pf}} = N_{\text{nf}} + N_{\text{mod}}$  as the vector of  $N_{\text{mod}}$  modal coefficients on the DtN boundary and  $N_{\text{nf}}$  desired near-field nodal values and shifting the system according to Eq. (3) yields

$$\mathbf{H}(\sigma_i) = \begin{bmatrix} \mathbf{D} \mathbf{F}^T \\ \mathbf{E}^T \end{bmatrix} \left[ \mathbf{K} - \kappa_0^2 \mathbf{M} + \mathbf{B}_1(\kappa_0) - \sigma_i \mathbf{M} + \mathbf{F} \mathbf{\Lambda} \mathbf{F}^T \right]^{-1} \mathbf{f}. \quad (14)$$

The inverse in Eq. (14) exhibits a very complicated dependency on the wave number, due to  $\mathbf{F} \mathbf{\Lambda} \mathbf{F}^T$ . To make this equation accessible for our methodology as in the interior case, we simplify the frequency dependence of the inverse of the system matrix. In the following we denote the frequency-independent part as  $\mathbf{A}_0(\kappa_0) = \mathbf{K} - \kappa_0^2 \mathbf{M} + \mathbf{B}_1(\kappa_0)$ . Since the matrices without the  $\mathbf{B}_1$  operator incorporate resonances at the eigenfrequencies of the interior problem, the nonsingularity of  $\mathbf{A}_0$  can not be guaranteed over the complete frequency range. To guarantee nonsingularity of the matrix at any frequency the  $\mathbf{B}_1$  condition was introduced. Next, define  $\mathbf{A}_{\sigma_i} = \mathbf{A}_0 - \sigma_i \mathbf{M}$  and apply the Sherman-Morrison-Woodbury formula (Golub and Van Loan, 1983) to express the inverse of Eq. (14) in terms of  $\mathbf{A}_{\sigma_i}^{-1}$ , which yields for  $\mathbf{H}$ ,

$$\mathbf{H}(\sigma_i) = \begin{bmatrix} \mathbf{D} \mathbf{F}^T \\ \mathbf{E}^T \end{bmatrix} \left[ \mathbf{A}_{\sigma_i}^{-1} - \mathbf{A}_{\sigma_i}^{-1} \mathbf{F} (\mathbf{\Lambda}^{-1} + \mathbf{F}^T \mathbf{A}_{\sigma_i}^{-1} \mathbf{F})^{-1} \mathbf{F}^T \mathbf{A}_{\sigma_i}^{-1} \right] \mathbf{f}, \quad (15)$$

that exhibits a simplified formulation in the sense that the inverse operator acts now only on the matrix  $\mathbf{A}_{\sigma_i}$ . Note that  $\mathbf{\Lambda}$  is a diagonal matrix, hence the inversion imposes no extra cost.

However, Eq. (15) does not exhibit the desired standard shifted structure of Eq. (4). To further simplify the problem, observe that several repeated terms can be identified in Eq. (15). Extracting these terms and rearranging them in a new matrix  $\mathbf{W}(\sigma_i) \in \mathbb{C}^{N_{\text{pf}} \times (N_{\text{mod}}+1)}$  yields

$$\mathbf{W}(\sigma_i) = \begin{bmatrix} \mathbf{w}_F & \mathbf{W}_F \\ \mathbf{w}_E & \mathbf{W}_E \end{bmatrix} = \begin{bmatrix} \mathbf{F}^T \\ \mathbf{E}^T \end{bmatrix} [\mathbf{I} - \sigma_i \mathbf{A}_0^{-1} \mathbf{M}]^{-1} \mathbf{A}_0^{-1} [\mathbf{f} \quad \mathbf{F}] \equiv \mathbf{L}^T [\mathbf{I} - \sigma_i \mathbf{A}]^{-1} \mathbf{R}, \quad (16)$$

where  $\mathbf{L} = [\mathbf{F} \quad \mathbf{E}]$ ,  $\mathbf{A} = \mathbf{A}_0^{-1} \mathbf{M}$ , and  $\mathbf{R} = \mathbf{A}_0^{-1} [\mathbf{f} \quad \mathbf{F}]$ . Note that the constant matrix has been already pulled out of the inverse operator. Since we allow unsymmetric matrices no symmetric decomposition has been applied. Equation (16) states the *unsymmetric standard shifted form* of Eq. (4), which is required to apply the block unsymmetric Lanczos process. Recall that the number of terms in the series for  $p(R, \theta)$  in Eq. (12) must equal the number of DtN terms,  $N_{\text{DtN}}$ . This restriction is apparent in Eq. (15), since otherwise the terms  $\mathbf{w}_F$  and  $\mathbf{W}_F$  could not be identified as shown. The left and right block starting vectors are  $\mathbf{L} \in \mathbb{C}^{N \times N_{\text{pf}}}$  and  $\mathbf{R} \in \mathbb{C}^{N \times (N_{\text{mod}}+1)}$ , where the system matrix is  $\mathbf{A} \in \mathbb{C}^{N \times N}$ . The formulation becomes unsymmetric because  $\mathbf{L} \neq \mathbf{R}$ . Moreover, note that no symmetric decomposition of  $\mathbf{A}_0$  is computed since the formulation is inherently unsymmetric.

## NUMERICAL EXAMPLE

In this subsection, the main question of the gain in efficiency of the proposed multi-frequency method in comparison to an iterative solver is illustrated.

The numerical problem is depicted in Fig. 1. A submarine-like structure determined by a dimensionless length parameter  $a = 5$  is immersed in an acoustic medium. Neumann boundary conditions are prescribed on the circular back portion of the structure as shown. The truncating DtN surface is a concentric circle of radius  $R = 7$ . It is one of the advantages of the exact DtN condition that the truncating surface can be very close to the sound source, hence decreasing the effort for discretization. A frequency range of  $\kappa a \in [0, 40]$  is chosen. The finite element discretization employs a mesh size  $h = 0.04$  providing a minimum of 20 elements per wavelength. The total number of degrees of freedom is  $N = 58793$ . The mesh consists of 58188 bilinear quadrilateral elements in the domain and 500 linear 1D elements on the DtN boundary. The DtN series is truncated at  $N_{\text{DtN}} = 10$  terms.

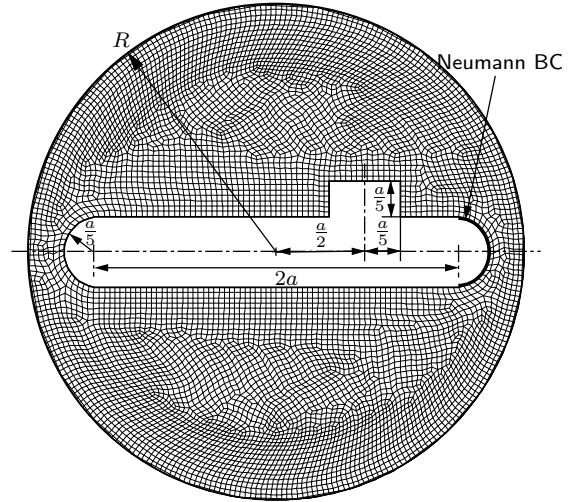


Figure 1: Submarine-like structure

In Fig. 2 the total computational times in seconds with the multiple-frequency solver vs. the number

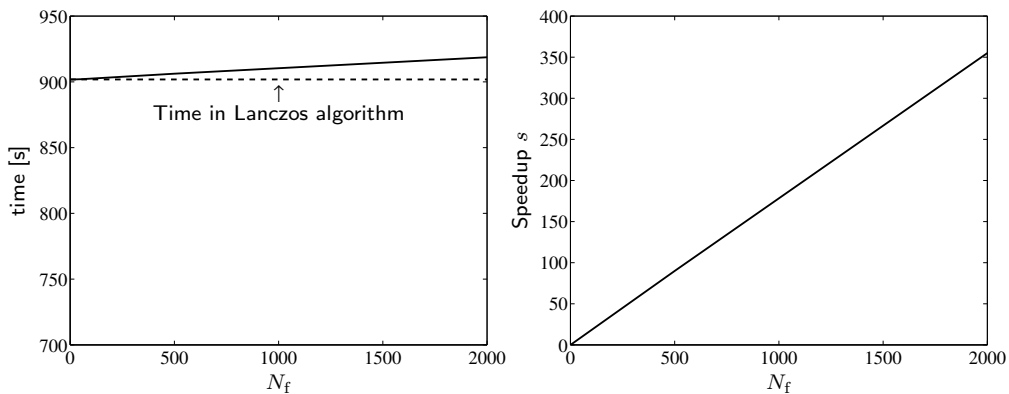


Figure 2: Total computation times and speedup  $s$  with the multiple-frequency solver.

of frequencies in the frequency window are shown for a number of Lanczos iterations  $n = 147$  and an expansion frequency  $\kappa_0 a = 20$ . The computations are carried out on a Pentium III processor with 850 MHz and 1 GB of core RAM. The computation time consists of two parts. There is a constant time that is spent independent of the number of frequencies evaluated in the factorization of  $\mathbf{A}_0$  and in the computation of the Krylov subspace projection, respectively. The second portion increases with the number of frequencies and is the time used in the remaining frequency loop. Even for the largest number of 2000 frequencies the portion spent in the loop is lower than 2%. Hence, the multiple-frequency solver is basically independent of the number of frequencies of interest.

In Fig. 2 on the right the multiple-frequency solver is compared with the solution times of a quasi-minimal residual (QMR) solver that employs an SSOR preconditioner tailored specifically for the case of exterior acoustics. The residual tolerance is set to  $10^{-6}$  and convergence is obtained in this example typically after several hundred iterations, depending on the frequency point of solution. As a measure, the speedup  $s$  is depicted, which is defined as  $s = \frac{t_{\text{SF}} \cdot N_f}{t_{\text{MF}}}$ , where  $t_{\text{SF}}$  is the solution time of the QMR solver. Note that the QMR solver times vary over the frequency window due to the frequency dependent spectrum of the matrix. Hence, a mean value is chosen to allow for comparison.

*The speedup is of several orders of magnitude, illustrating the significant increase in computational efficiency by the proposed multiple-frequency solver.*

## CONCLUSION

An efficient algorithm for the simultaneous solution of the Helmholtz equation at multiple frequencies over a restricted part of the complete computational domain is presented. We treat the general case of an unsymmetric system appearing in the treatment of both, far-field points and selected points in the near-field of the sound source. The proposed method reformulates the matrix problem into a standard shifted form which is projected on a Krylov subspace of much smaller dimension than the system size. This subspace is obtained by applying an unsymmetric block Lanczos algorithm. The main costs of the algorithm are one factorization of  $\mathbf{A}_0$  and the evaluation of matrix-vector products with this matrix in the Lanczos algorithm. As compared to the full factorization of the system matrix for each frequency in a traditional direct solution approach, this accounts for the efficiency of the method. The additional effort for the solutions of the small banded block system Eq. (8) and the dense block system in the solution for  $\mathbf{q}$  at each frequency is negligible. The numerical experiments show the significant improvement in computational efficiency of the proposed method. The speedup for the solution in a frequency window can be several orders of magnitude for the system size considered here.

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