An ILU–type preconditioner for the Boundary Element Method

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Abstract

We are presenting an incomplete factorization preconditioner for the boundary integral formulation of the time harmonic Helmholtz equation in 3 D. Out of the matrix of the near field interactions of a Fast Boundary Element Method we are constructing the preconditioner using an incomplete LU-factorization provided in the *SPARSKIT V. 2* package by Saad. This preconditioner is tested and discussed for Restarted GMRes. The paper is dedicated to external problems only. These problems are solved by the hyper-singular formulation of Burton and Miller. The latter results in a non-compact operator. Especially in conjunction with non-smooth surfaces the preconditioner reduces the number of iterations needed by GMRes significantly. Examples that are investigated are a tire noise problem and a piston compressor.

1 Introduction

We consider only problems where we apply so–called Fast Boundary Element Methods. Namely, the Regular Grid Method (RGM), cf. [1], and the Multilevel Fast Multipole Method (MLFMA), cf. [2], are used to overcome the $\mathcal{O}(N^2)$ memory requirement of the standard BEM. These methods directly approximate the matrix–vector–product Au by

$$Au = (I - A_{\text{near}} - A_{\text{far}})u = (I - A_{\text{near}})u + v_{\text{far}}(u) \quad (1)$$

with a sparse matrix A_{near}. This matrix is calculated directly and stored using standard sparse matrix techniques. The vector v_{far} is evaluated directly without filling the dense matrix A_{far}. It is approximated either by utilizing the Fast Fourier Transformation or a Multipole Expansion.

To suppress the spurious frequencies, we are using the Burton/Miller approach [3] with a coupling parameter $\alpha = i/k$. The hyper-singular operator appearing in this approach is not any more compact. This is one of the reasons why iterative solvers do not converge well or even fail to converge at all.

This paper is organized as follows. In the next section, we define the problem we are interested to solve. Section three deals with the construction of a suitable preconditioner. This preconditioner is then applied to a numerical example in section three.

2 Problem

The main goal of our work is it to use finite element surface meshes of a structure and the nodal results of an FEM-simulation directly for the calculation of radiated or scattered sound field of such an object using the Boundary Element Method in the frequency domain. Hence, we like to solve the boundary integral representation of the Helmholtz equation

$$p(y) + \int_\Gamma k(x,y)p(x)\,d\Gamma \;=\; f(y) \tag{2}$$

or, equivalently,

$$(\mathcal{I} - \mathcal{A})p \;=\; f \tag{3}$$

with a given right hand side f representing surface loads. To suppress the spurious frequencies we are using the Burton/Miller approach. The appearing hyper-singular operator is a non-compact operator. Thus the eigenvalues of $\mathcal{A}$ are not clustered. This results in the fact that for the eigenvalue distribution of the matrix representing the discretization of the integral operator iterative solvers like GMRes are not well suited (see [4]). Further, we found that the iterative solvers are sensitive to non-smooth surfaces. But when using finite element surface meshes we have lots of edges and vertices on the surface. What may cause a further reduction of the convergence rate of iterative solver. To overcome these problems arising with the Burton/Miller approach, the application of the Fast Boundary Element Methods, and the fact that the surface Γ is not smooth we will apply a suitable preconditioner described in the following section.

3 Construction of the preconditioner

In the following, we want to construct a preconditioner for the (non-symmetric) dense linear system $(I - A)p = b$ which arise from the discretization of a hyper-singular integral operator. Thus we will solve the preconditioned system $(I - A)M^{-1}y = b$ and $p = M^{-1}y$. To make GMRes performing more efficiently, we want the operator $(\mathcal{I} - \mathcal{A})\mathcal{M}^{-1}$ to be compact. Further, a suitable preconditioner M should possess the following properties:

- only the entries of A_{near} are needed for its construction
- sparse representation of M^{-1} ($\mathcal{O}(N)$ entries)
- $(\mathcal{I} - \mathcal{A})\mathcal{M}^{-1}$ is compact.

There exist various ways of preconditioning for dense linear systems arising from the Boundary Element Method (see for example [5]). The method ideal matching our requirements among them is the Operator Splitting Preconditioner (OSP). It's construction is based on the fact that the product of a compact operator with a bounded operator is compact. Thus the operator

$$(\mathcal{I} - \mathcal{A})\mathcal{M}^{-1} = (\mathcal{I} - \mathcal{A}_{\text{near}} - \mathcal{A}_{\text{far}})\mathcal{M}^{-1} \tag{4}$$

is compact (plus an identity) if we chose

$$\mathcal{M} = \mathcal{I} - \mathcal{A}_{\text{near}} \tag{5}$$

as $\mathcal{A}_{\text{far}}$ is already compact because it is an integral operator with an continuous kernel.

This choice of the preconditioner satisfies all the criteria of suitable precondition for our problems if we use a sparse representation of the inverse of M. For this task some kind of incomplete LU-decomposition seems to be well suited. In detail, we apply the routines provided in the *SPARSKIT V. 2* package by Saad, namely, we are using a complex version of the $ilut(\tau, p)$ routine [6]. With the parameter τ the minimal relative size of an entry in the factors L and U is prescribed. The second parameter p controls the maximum number of fill–in elements per row. Hence, we can prescribe the memory requirement with the parameter p whereas the threshold parameter τ can influence the time for the calculation of the decomposition and its application in the iterative solution. We like to emphasize at this point that the time for evaluating a matrix–vector–product by the Fast Boundary Element Methods is dominated by calculating $v_{\text{far}}(u)$. The time for the evaluation of $A_{\text{near}}u$ and for solving $LUu = z$ is negligible since these are sparse matrix operations.

4 Numerical examples

We define the residual of the iterative solution at the n-th iteration step as

$$\frac{||(I - A)M^{-1}y_n - b||}{||b||} = \varepsilon_n \quad . \tag{6}$$

The iteration process is terminated if $\varepsilon_n < 10^{-6}$ is satisfied. The numerical

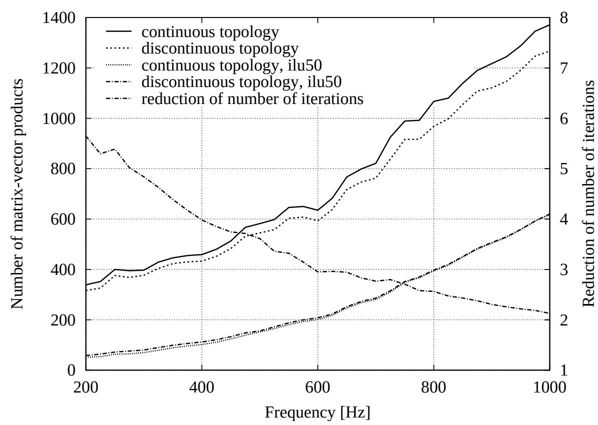

Figure 1: Sedan tire noise analysis, comparison of convergence in terms of frequency with and without preconditioning for iterative solver GMRes

example is the model of a car standing on a rigid ground. Two surface meshes were available- a mesh with continuous element topology having $\approx$ 40000 elements and a mesh with discontinuous topology with $\approx$ 25000 elements. We found that the number of iterations needed by the GMRes grows rapidly with increasing frequency. Now, we aim on reducing the costly part of the solution process, i.e. the iterative solution, by applying the $ilut(\tau, p)$ preconditioner.

We use a fill–in parameter of $p = 50$. With this choice the cost of the preconditioner in terms of memory requirement is approximately 15% of the total memory in use at highest frequency ($\approx$ 235 Mb). In Fig. 1 the reduction of number of iterations needed by the GMRes solver is shown.

The number of iterations is more then halved over the entire frequency range. The expenditure of preconditioner calculation and its application in the iterative solution process is negligible. So, the total solution time is also halved.

In the unpreconditioned version, the larger model of continuous topology (39502 elements) requires about 50 to 100 iterations more than the smaller model of discontinuous topology (25810 elements). If preconditioning is applied we observe hardly any differences in convergence between both discretizations.

Concerning this tire noise example, we can summarize that performance of GMRes is remarkably affected by the $ilut(\tau, p)$ preconditioner. Speedups between two and five compared to unpreconditioned solution are reported.

5 Conclusion

The Incomplete LU–Decomposition of the matrix representing the near field interactions of a Fast Boundary Element Method is well suited as a preconditioner for exterior acoustic problems. It significantly reduces the number of iterations needed by the iterative solvers investigated. The extra time for calculation and application of the preconditioner is negligible since the time for evaluation of Au is dominated by the evaluation of $v_{\text{far}}(u)$.

Especially for problems with highly non-smooth surfaces the usage of a preconditioner is essential as Restarted GMRes do not converge at all in the unpreconditioned cases. Full GMRes converges slowly if no preconditioning is applied.

The preconditioned GMRes performed the best in the low- to mid-frequency range as long as no or only a few restarts occur. Thus, a good balance of the memory distribution between the preconditioner and the basis for the Krylov Space must be found. In general, a value of $p = 10 \ldots 20$ for the fill-in parameter leads to the most efficient results. Apparently, the presented preconditioner requires additional memory. However, a single iteration step itself is very costly. Therefore, a reduction of the number of iterations is often more interesting than the gain of some computer memory.

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