FAST SOLUTIONS OF BOUNDARY ELEMENT SOLUTIONS OF THE EXTERIOR HELMHOLTZ EQUATION

PACS: 43.20.Fn, 43.28.Js, 43.30.Jx

S. Amini

Mathematics; University of Salford

M5 4WT; UK

Tel: 00 44 161 2955353

Fax: 00 44 161 2955559

Email: s.amini@salford.ac.uk

Abstract

In this paper we study the multi-level fast multipole solution of the boundary integral formulation for the Helmholtz equation exterior to a closed curve in 2D. We employ the uniquely solvable formulation due to Burton and Miller, involving the hypersingular operator, the normal derivative of the double layer potential. We state an outline of the derivation of error and $\mathcal{O}\left(n \ln^2 n\right)$ complexity analysis of the method. We present some numerical results which seem to support the theory.

1 Background

In Boundary Element Methods (BEM), the linear partial differential equation of interest is reformulated as a Boundary Integral Equation (BIE), which is then discretised using Finite Element (FE) techniques. Boundary element methods partition the boundary into n elements, generally resulting in an $n \times n$ system of linear equations. Gaussian elimination type solvers require $\mathcal{O}\left(n^3\right)$ arithmetic operations. In general, the use of an iterative solver, possibly with suitable preconditioners, results in an $\mathcal{O}\left(n^2\right)$ solution. However, these methods cannot improve on this complexity estimate, since simply forming the coefficient matrix requires $\mathcal{O}\left(n^2\right)$ arithmetic operations.

The so-called 'fast methods', such as wavelet and multipole expansions and panel clustering, improve on the $\mathcal{O}\left(n^2\right)$ complexity estimate. The whole coefficient matrix is not formed. In its place, a sparse approximation or a sparse approximate decomposition is used. The fast methods then employ an iterative solver which can be implemented using only matrix–vector multiplication, without needing to have access to the full matrix, which in general isn't available. Here, we use a Galerkin approximation with fast multipole exapnsions and employ an iterative solver from the Krylov subspace class.

Denote by Ω_+, the exterior of the closed curve Γ. The field of time-harmonic acoustic scattering from Γ_+ is described by the Helmholtz equation with wavenumber $k = \omega/c$, where ω is the frequency and c the sound speed in Ω_+ . We consider the case of the Neumann boundary condition, relating to sound hard structures, and for uniqueness impose a suitable condition on ϕ 'at infinity', such as the Sommerfeld radiation condition. This problem now has a unique solution as stated in the theorem below [5, Thms 7.6.1,7.6.2].

Theorem 1.1. *Let $k > 0$. Then, for $s \geqslant -\frac{1}{2}$, the exterior Neumann problem*

$$\left.\begin{aligned} (\nabla^2 + k^2)\phi(\mathbf{p}) = 0, \qquad \mathbf{p} \in \Omega_+ \\ \frac{\partial \phi}{\partial \mathbf{n}}(\mathbf{p}) = f(\mathbf{p}) \in \mathcal{H}^s(\Gamma) \\ \lim_{r\to\infty} r^{\frac{1}{2}} \left(\frac{\partial \phi}{\partial r} - \mathrm{i}k\phi\right) = 0 \end{aligned}\right\}$$

has a unique solution $\phi \in \mathcal{H}^{s+3/2}_{\mathrm{loc}}(\Omega_+)$.

For definitions of the Sobolev spaces $\mathcal{H}^s(\Gamma)$ and $\mathcal{H}^s_{\mathrm{loc}}(\Omega_+)$ see [5, 6]. The free space Green's function, also known as the fundamental solution, for the two-dimensional Helmholtz equation is

$$G_k(\mathbf{p}, \mathbf{q}) = \frac{\mathrm{i}}{4} H_0(k\|\mathbf{p} - \mathbf{q}\|),$$

where H_0 is the Hankel function of the first kind and order zero and $\|\mathbf{p}-\mathbf{q}\|$ is the distance between field or boundary points $\mathbf{p}$ and $\mathbf{q}$. Here, we employ the uniquely solvable direct boundary integral reformulation of the Helmholtz equation due to Burton and Miller [4]

$$\left(-\frac{1}{2}\mathcal{I} + \mathcal{M}_k + \mathrm{i}\gamma\mathcal{N}_k\right)\phi = \left(\mathcal{L}_k + \mathrm{i}\gamma\left(\frac{1}{2}\mathcal{I} + M_k^T\right)\right)\frac{\partial \phi}{\partial \mathbf{n}}, \quad \mathbf{p} \in \Gamma, \tag{1.1}$$

where γ is a positive real coupling parameter and the operators are defined as:

$$(\mathcal{L}_k u)(\mathbf{p}) = \int_\Gamma G_k(\mathbf{p},\mathbf{q}) u(\mathbf{q})\ d\Gamma_q, \qquad (\mathcal{M}_k^T u)(\mathbf{p}) = \frac{\partial}{\partial \mathbf{n}_p} \int_\Gamma G_k(\mathbf{p},\mathbf{q}) u(\mathbf{q})\ d\Gamma_q,$$
$$(\mathcal{M}_k u)(\mathbf{p}) = \int_\Gamma \frac{\partial G_k}{\partial \mathbf{n}_q}(\mathbf{p},\mathbf{q}) u(\mathbf{q})\ d\Gamma_q, \qquad (\mathcal{N}_k u)(\mathbf{p}) = \frac{\partial}{\partial \mathbf{n}_p} \int_\Gamma \frac{\partial G_k}{\partial \mathbf{n}_q}(\mathbf{p},\mathbf{q}) u(\mathbf{q})\ d\Gamma_q.$$

In general we choose $\gamma = \frac{1}{k}$ to improve the conditioning of the equation (1.1), as suggested in [2]. For the Neumann problem, equation (1.1), viewed for convenience in shorthand notation $\mathcal{A}\phi = g$, is first solved to find ϕ on Γ. Once the Cauchy data $(\phi, \partial\phi/\partial\mathbf{n})$ on Γ is known the solution to our exterior problem can be found using the Helmholtz representation formula

$$\phi(\mathbf{p}) = \int_\Gamma \frac{\partial G_k}{\partial \mathbf{n}_q}(\mathbf{p},\mathbf{q}) \phi(\mathbf{q})\ d\Gamma_q - \int_\Gamma G_k(\mathbf{p},\mathbf{q}) \frac{\partial \phi}{\partial \mathbf{n}_q} d\Gamma_q, \qquad \mathbf{p} \in \Gamma_+.$$

2 Multipole Expansion

At the heart of multipole algorithms is the approximation of the kernel of the boundary integral operator $\mathcal{A}$, by a *separable* or *degenerate* kernel of the form:

$$K(\mathbf{p},\mathbf{q}) \approx \sum_{l,m=1}^{L} f_l(\mathbf{p}) b_{lm} g_m(\mathbf{q}) = f(\mathbf{p})^T B g(\mathbf{q}) \tag{2.2}$$

where f and g are $L-$vectors and B is an $L \times L$ matrix independent of $(\mathbf{p},\mathbf{q})$. If the separable approximation (2.2) was valid everywhere, then it is straightforward to show that the $n \times n$ boundary element matrix A, with $A_{i,j} =< \mathcal{A}\psi_i, \psi_j >$, where $\{\psi_i\}$ are the Galerkin basis functions, could be approximated by a rank L decomposition UBV, where U is an $n \times L$ matrix and V and $L \times n$. Therefore, the formation of matrix-vector product $A\mathbf{x}$, essential at every step of the iterative solution of the boundary element discretisation can be approximated by $U(B(V\mathbf{x}))$ at the cost of $2nL + L^2$ operations in place of n^2. Note that it is not necessary to form elements of A, only those of U, V and B.

Unfortunately, for problems with Green's function type kernels, which are singular at $\mathbf{p} = \mathbf{q}$, approximate separable expansions of the form (2.2), with small to moderate expansion length L, can only be found provided $\mathbf{p}$ and $\mathbf{q}$ are in far-away clusters.

Central to the theory behind the fast multipole methods for the Helmholtz equation is Graf's addition theorem [1, (9.1.79)], which is used in the derivation of kernel expansion for the 'basic' kernel of $\mathcal{L}_k$, all others following from this; see [7].

Theorem 2.1. *Let L be an odd integer. For $\mathbf{p}, \mathbf{c}, \mathbf{q}, \mathbf{d} \in \mathbb{R}^2$ with $\|(\mathbf{q}-\mathbf{d})-(\mathbf{p}-\mathbf{c})\| < \|\mathbf{c}-\mathbf{d}\|$, and k a positive real $\frac{\mathrm{i}}{4}H_0(\|k(\mathbf{p}-\mathbf{q})\|) = f^T B g + R_L$, where f and g are the L-vectors*

$$\begin{pmatrix} (-1)^M J_M(k\|\mathbf{p}-\mathbf{c}\|)e^{-iM\theta_{p-c}} \\ \vdots \\ -J_1(k\|\mathbf{p}-\mathbf{c}\|)e^{-i\theta_{p-c}} \\ J_0(k\|\mathbf{p}-\mathbf{c}\|) \\ J_1(k\|\mathbf{p}-\mathbf{c}\|)e^{i\theta_{p-c}} \\ \vdots \\ J_M(k\|\mathbf{p}-\mathbf{c}\|)e^{iM\theta_{p-c}} \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} (-1)^M J_M(k\|\mathbf{q}-\mathbf{d}\|)e^{iM\theta_{q-d}} \\ \vdots \\ -J_1(k\|\mathbf{q}-\mathbf{d}\|)e^{i\theta_{q-d}} \\ J_0(k\|\mathbf{q}-\mathbf{d}\|) \\ J_1(k\|\mathbf{q}-\mathbf{d}\|)e^{-i\theta_{q-d}} \\ \vdots \\ J_M(k\|\mathbf{q}-\mathbf{d}\|)e^{-iM\theta_{q-d}} \end{pmatrix},$$

respectively, with M defined by $L = 2M+1$. B is the $L \times L$ Toeplitz matrix

$$\frac{\mathrm{i}}{4}\begin{pmatrix} H_0 & H_1\omega & H_2\omega^2 & \cdots & H_M\omega^M & 0 & \cdots & 0 \\ -H_1\omega^{-1} & H_0 & H_1\omega & \ddots & & \ddots & \ddots & \vdots \\ H_2\omega^{-2} & -H_1\omega^{-1} & H_0 & & & & & \\ \vdots & & & & & & & \\ (-1)^M H_M\omega^{-M} & & & & & & & \\ 0 & \ddots & & & & & & \\ \vdots & & & & & & \ddots & H_1\omega \\ 0 & & & & & & -H_1\omega^{-1} & H_0 \end{pmatrix}$$

in which we have written H_l for $H_l(k\|\mathbf{c}-\mathbf{d}\|)$ and ω for $e^{i\theta_{c-d}}$. The remainder term is given by

$$R_L = \sum_{|m|>M} H_m(x)e^{\mathrm{i}m\theta_x}J_m(y)e^{-\mathrm{i}m\theta_y} + \sum_{|m|\leqslant M} H_m(x)e^{\mathrm{i}m\theta_x} \sum_{|l|>M \ or \ |l+m|>M} J_{m+l}(u)e^{-\mathrm{i}(m+l)\theta_u}J_l(v)e^{\mathrm{i}l\theta_v}.$$

with $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{v}$ defined by $\mathbf{x} = k(\mathbf{c}-\mathbf{d})$, $\mathbf{u} = k(\mathbf{q}-\mathbf{d})$, $\mathbf{v} = k(\mathbf{p}-\mathbf{c})$, $\mathbf{y} = k(\mathbf{q}-\mathbf{d})-k(\mathbf{p}-\mathbf{c})$.

It is proved rigorously in [7], using theorem 2.1, that for the practical case where $kh \sim 1$, multilevel fast multipole algorithms can solve the problem to within the level of discretisation error in only $\mathcal{O}\left(n \log^2 n\right)$ arithmetic operations.

3 Numerical Experiments

We use the multi-level fast multipole algorithm to solve the exterior Helmholtz equation with Neumann boundary conditions for large values of k and hence large n. We use the hypersingular Burton and Miller reformulation (1.1), discretised by a Galerkin method with piecewise linear basis functions. The boundary Γ is taken to be an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with length 2π and with $b = 2a$. The test problem was generated by placing a point source of unit strength at the point $\mathbf{p} = (0.0, 0.5)$ within Γ. The iterative solver GMRES was used for the solution of the boundary element equations, to within the level of discretisation error; see also [8, 3]. The results are summarised in Table 1. For $n > 2^{10}$, the CPU time and memory requirements for the standard method were estimated since they were beyond the capabilities of our computer. For example, for $n = 2^{15}$, the estimated 2^{21} seconds of CPU time is 582 hours, and the estimated memory required is 17 Gigabytes. As can be seen from Table 1, use of the fast method does not add to the discretisation error of the boundary element method. However, CPU time and memory requirements are reduced from $\mathcal{O}\left(n^2\right)$ to almost $\mathcal{O}\left(n\right)$. This is consistent with the predicted $\mathcal{O}\left(n \log^2 n\right)$ complexity estimate.

References

[1] M. Abramowitz and I. Stegun, *Handbook of mathematical functions*, Dover, New York, 1974.

[2] S. Amini, *On the choice of the coupling parameter in boundary integral formulations of the exterior acoustic problem*, Appl. Anal. **35** (1990), 75–92.

[3] ———, *On boundary integral operators for the Laplace and the Helmholtz equations and their discretisations*, Engineering Analysis with Boundary Elements **23** (1999), 327–337.

k	n	$\|\phi - \hat{u}_n\|_{\mathcal{L}^2}$		CPU time		Memory	
		Standard	FMM	Standard	FMM	Standard	FMM
2^2	2^6	$2^{-7.21}$	$2^{-7.21}$	$2^{3.04}$	$2^{2.57}$	$2^{12.00}$	$2^{13.18}$
2^3	2^7	$2^{-7.36}$	$2^{-7.36}$	$2^{5.01}$	$2^{4.52}$	$2^{14.00}$	$2^{14.53}$
2^4	2^8	$2^{-7.51}$	$2^{-7.51}$	$2^{7.02}$	$2^{5.94}$	$2^{16.00}$	$2^{15.70}$
2^5	2^9	$2^{-7.54}$	$2^{-7.54}$	$2^{9.02}$	$2^{7.26}$	$2^{18.00}$	$2^{16.74}$
2^6	2^{10}	$2^{-7.56}$	$2^{-7.56}$	$2^{11.02}$	$2^{8.22}$	$2^{20.00}$	$2^{17.78}$
2^7	2^{11}		$2^{-7.56}$	$2^{13.02}$	$2^{9.26}$	$2^{22.00}$	$2^{18.83}$
2^8	2^{12}		$2^{-7.56}$	$2^{15.02}$	$2^{10.28}$	$2^{24.00}$	$2^{19.95}$
2^9	2^{13}		$2^{-7.56}$	$2^{17.02}$	$2^{11.29}$	$2^{26.00}$	$2^{20.99}$
2^{10}	2^{14}		$2^{-7.56}$	$2^{19.02}$	$2^{12.30}$	$2^{28.00}$	$2^{22.02}$
2^{11}	2^{15}		$2^{-7.56}$	$2^{21.02}$	$2^{13.31}$	$2^{30.00}$	$2^{23.09}$

Table 1: Multi-level FMM for the Helmholtz equation

[4] A. J. Burton and G. F. Miller, *The application of integral methods for the numerical solution of boundary value problems*, Proceedings of the Royal Society of London **A232** (1971), 201–210.

[5] G. Chen and J. Zhou, *Boundary element methods*, Academic Press, London, 1992.

[6] D. L. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley, New York, 1983.

[7] A. T. J. Profit, *On fast multipole methods for the solution of of the Helmholtz problem*, Ph.D. thesis, Dept. of Maths., University of Salford, Salford, UK., 1999.

[8] S. Amini and N. D. Maines, *Preconditioned Krylov subspace methods for the boundary integral solution of the helmholtz problem*, International Journal of Numerical Methods in Engineering **41** (1998), 875–898.