Adaptive multigrid for robust and efficient sound field simulations

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Abstract

When using traditional acoustic finite element packages, one has to provide a mesh which has to be proven to be sufficient for representing the solution to be calculated. Since there is no a priori knowledge about the solution in general, the quality of the solution can not be guaranteed by this approach.

For a robust and efficient sound field simulation, we use the combination of the truncated modal basis approach, adaptive mesh refinements and fast multigrid methods. For the modal basis approach, it is known that the representation of sound fields by modal basis functions is optimal with respect to the L2 error norm. So, it is necessary to have a finite element basis which minimizes the discretisation error when computing the modal basis. We reach this goal by applying adaptive mesh refinements. Additionally, this yields the opportunity of using fast multigrid methods.

1 Introduction

A typical problem using finite element acoustic solvers is that the user has to provide a mesh which is optimal with respect to the solution to be achieved. Since there is no a priori knowledge about the field distribution, a common rule of thumb is used, which says that for acoustic problems a mesh consisting of at least six nodes per wavelength in each spatial direction is sufficient to represent the computed solution.

The drawback of this rule is the same as with many other rules of thumb: The validity depends on a set of assumptions which have to be studied before solving the problem. One assumptions is that the user is only interested in having a small global error, such as described by the RMS-error. This approach can be very dangerous as we'll show later on.

2 Subspace methods for room acoustic

For room acoustics, the need for a finite element calculation is finally limited by the frequency where the mode density is sufficient for the use of particle based models. This limitation has one interesting aspect for the work: The truncated modal basis approach is a very efficient way to solve the problem. Once having a modal basis, one can easily make parametric studies for different boundary conditions and different source positions. This is due to the fact that the projection of the original Helmholtz equation onto a truncated modal basis results in small systems of equations which can be solved quickly [2].

For the final simulation result, each mode has to be weighted and superimposed to the other ones. It should be noted that every mode contributes to the sound field for all possible frequencies. So, the simulation quality depends on the number of modes to be calculated. On the other hand, the influence of a mode decreases with increasing distance between its eigenfrequency and the considered frequency of interest. This discussion is not in the focus of this paper. Our task was to compute the m first eigenmodes numerically within given error bounds.

Interior acoustics simulation based on wave theory means to find solutions of the boundary value problem [4]:

$$\triangle p + k^2 p = 0 \quad \text{in} \quad \Omega \tag{1}$$

$$\frac{\partial p}{\partial n} + i\,k\,\rho_0\,c\,v_n = 0 \quad \text{on} \quad \Gamma_1 \tag{2}$$

$$\frac{\partial p}{\partial n} + i\,k\,\rho_0\,c\,A_n\,p = 0 \quad \text{on} \quad \Gamma_2 \tag{3}$$

$$\text{and} \quad \int_\Omega p\,d\Omega = 0 \tag{4}$$

were k is the wave number as a parameter, Ω is the field-domain and $\Gamma = \Gamma_1 + \Gamma_2$ its boundary. ρ_0 denotes the mass density and c the sound velocity of the fluid. A_n is the wall-admittance and v_n is the normal velocity as the driving force.

After transforming the problem (1) to (4) into its weak formulation and making a Galerkin ansatz, the problem reduces to a m-dimensional system of linear equations:

$$(S + i\,k\,C - k^2 M)\,x = -i\,k\,f \tag{5}$$

where S is the so called stiffness matrix, C the damping matrix, M the mass matrix, x the coefficient vector and f the acoustic force vector. Due to the orthogonality of the eigenfunctions, the approximation error $\epsilon_m = p - p_m$ takes its minimum in the mean square. Thus, the quality of our sound field calculation only depends on the number of eigenmodes building the modal space.

3 Rectangular room as a simple test case

The rectangular room can be seen as perfect test case for acoustic solvers, since the analytic solution is well known. In case of rigid walls ($A_n = 0$) and the absence of any driving force ($v_n = 0$) the problem (1) to (4) reduces to

$$\triangle p + k^2 p = 0 \quad \text{in} \quad \Omega\,, \tag{6}$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \Gamma \tag{7}$$

or the matrix eigenvalue problem

$$(S - k^2 M)\,x = 0 \tag{8}$$

respectively,

For the rectangular room, the modes can be determined by

$$\frac{\varphi_{lmn}}{\sqrt{X}} = \cos\left(\frac{l\pi x}{L_x}\right)\cos\left(\frac{m\pi y}{L_y}\right)\cos\left(\frac{n\pi z}{L_z}\right)\,. \tag{9}$$

Here, L_x, L_y and L_z are the physical dimensions of the room and X is a standardization constant which can be computed by

$$X = \frac{1}{L_x\,L_y\,L_z} \cdot \begin{cases} 1: & l = m = n = 0 \\ 2: & \text{exactly one index } \neq 0 \\ 4: & \text{exactly two indices } \neq 0 \\ 8: & \text{all indices } \neq 0 \end{cases}\,. \tag{10}$$

The eigenfrequencies f_{lmn} of the rectangular room can be computed analytically by

$$f_{lmn} = \frac{1}{2}\,c_0\,\sqrt{\frac{l^2}{{L_x}^2} + \frac{m^2}{{L_y}^2} + \frac{n^2}{{L_z}^2}}\,. \tag{11}$$

3.1 RMS-error

First, we computed the RMS[1]-error.

$$||E_p||_2 := \sqrt{\sum_{i=0}^{N-1} (\tilde{p}_i - p_i)^2} \quad (12)$$

for a hard banded room with the physical dimension $3m \times 5m \times 7m$ on different discretisation levels.

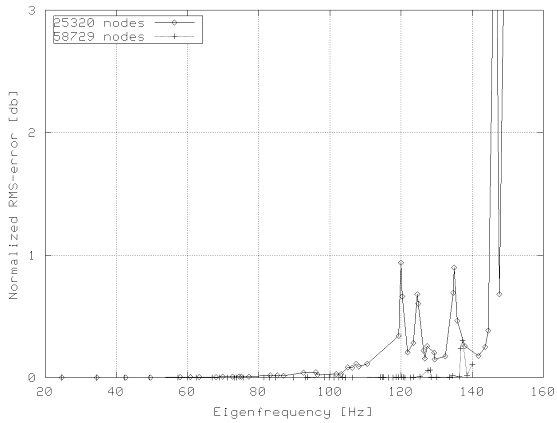

Figure 1: RMS error

Here, $\tilde{p}_i$ denotes the computed and p_i the real sound pressure level at the mesh node i. N is the number of nodes within the mesh.

To not cause a growth of the error when using a finer mesh with more nodes N, the use of the normalized RMS-error instead of (12) is recommended:

$$||E||_2 := \sqrt{\frac{\sum_{i=0}^{N-1} (\tilde{p}_i - p_i)^2}{\sum_{i=0}^{N-1} p_i^2}} \quad (13)$$

The reader may note that the graph labeled '58729 nodes' is the result of a simulation on mesh which approximately is conforming to the 'six nodes per wavelength'-rule for 140 Hz. As expected, the error below this frequency is small ($||E||_2 \ll 1dB$).

3.2 Maximum error

Now we computed the maximum error within the same room and based on the same discretisations as used in section 3.1 by using the normalized maximum error

$$||E||_\infty := \frac{N}{\sqrt{\sum_{i=0}^{N-1} p_i^2}} \max_{i \in \{0 \ldots N-1\}} |\tilde{p}_i - p_i| \quad (14)$$

[1]RMS:root mean square

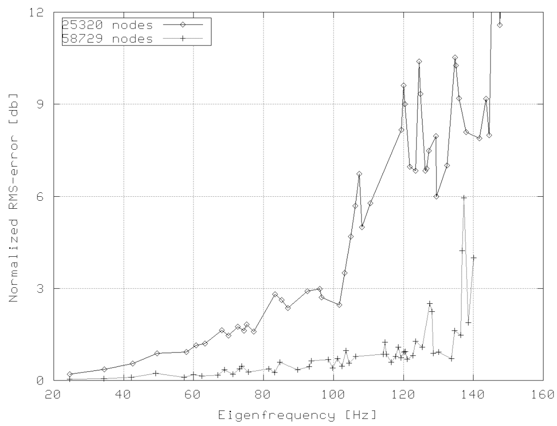

Figure 2: Maximum error

The result of applying this error norm on the solution is visualized in figure 2.

It can be stated that the normalized maximum error arises to large values much faster than the RMS error. One can easily check, that the maximum error exceeds the 1dB limit already at frequencies which are beyond the frequencies denoted by the rule of thumb.

3.3 Consequences

As a consequence of the results in this test case, one should use the 'six nodes per wavelength'-rule only for simple test cases where the solution is smooth enough to be represented on such a mesh. For more realistic problems, e.g. rooms with rough edges and/or interiors which are connected by alleyways, we strongly suggest to use adaptive generated meshes.

4 Adaptive multigrid

We treat the number of desired eigenmodes as a fixed number and try to compute these eigensolutions by a finite element method within given error bounds.

Since we cannot compare our approximations with the unknown true solutions, we replace the latter ones by "better" approximations. In our case, we solve the eigenvalue problem (8) in finite element spaces $\mathcal{S}_L \subset H^1(\Omega)$ spanned by piecewise defined polynomials of linear order. To estimate the discretisation error, we compare these solutions with those from an approximation of quadratic order. We construct these higher order finite element spaces $\mathcal{S}_Q \supset \mathcal{S}_L$ by a hierarchical extension with piecewise quadratic polynomials associated to the midnodes of edges in the underlying triangulation abbreviated as

$$\mathcal{S}_Q = \mathcal{S}_L \oplus \mathcal{Q}\,. \tag{15}$$

If we denote $\{\varphi_{L,i} \mid i = 1, \ldots, n_L\}$ as the FE-basis of $\mathcal{S}_L$ and $\{\varphi_{Q,i} \mid i = 1, \ldots, n_Q\}$ as the FE-basis of $\mathcal{Q}$, the corresponding discrete eigenvalue problem can be written in block form:

$$\begin{aligned} &\begin{pmatrix} S_{LL} & S_{LQ} \\ S_{QL} & S_{QQ} \end{pmatrix} \begin{pmatrix} \varphi_{L,i} \\ \varphi_{Q,i} \end{pmatrix} \\ &= \lambda_{\mathcal{S}_Q,i} \begin{pmatrix} M_{LL} & M_{LQ} \\ M_{QL} & M_{QQ} \end{pmatrix} \begin{pmatrix} \varphi_{L,i} \\ \varphi_{Q,i} \end{pmatrix} \end{aligned} \tag{16}$$

As described in [3], we can estimate the total error $\tilde{\lambda}_i - \lambda_i$ by an estimation of the iteration error $\tilde{\lambda}_i - \lambda_{\mathcal{S}_Q,i}$:

$$\tilde{\lambda}_i - \lambda_i \leq \frac{1}{1-\beta}(\tilde{\lambda}_i - \lambda_{\mathcal{S}_Q,i}) \ . \tag{17}$$

The estimate of the iteration error is [5]:

$$\tilde{\lambda}_i - \lambda_{\mathcal{S}_Q,i} \leq \frac{\lambda_{i+1}}{\lambda_{i+1} - \tilde{\lambda}_i} \frac{(r_i, B^{-1} r_i)}{(1-\gamma_1)} \ . \tag{18}$$

Here, B is a preconditioner of the matrix S and is constructed as

$$B := \begin{pmatrix} B_{LL} & 0 \\ 0 & D_{QQ} \end{pmatrix} \tag{19}$$

B_{LL} is the multigrid preconditioner described in [6] and D_{QQ} the diagonal part of S_{QQ}.

The local error components as described in [1] can be quantified and initiate a local refinement of the mesh.

Thus, we construct a sequence of finite element spaces

$$\mathcal{S}^k \subset \mathcal{S}^{k+1} \subset \ldots \subset H^1(\Omega) \tag{20}$$

until the approximate solutions of (6),(7) reach the desired accuracy.

5 Results

As a remarkable result, we found out that the eigenfrequency calculation for a rectangular room with two different meshes, but with approximately the same dimension, is much more precise if one uses adaptive generated meshes. In figure 3 this result is visualized based on a calculation of the first 25 eigenfrequencies of the test room described in section 3.1.

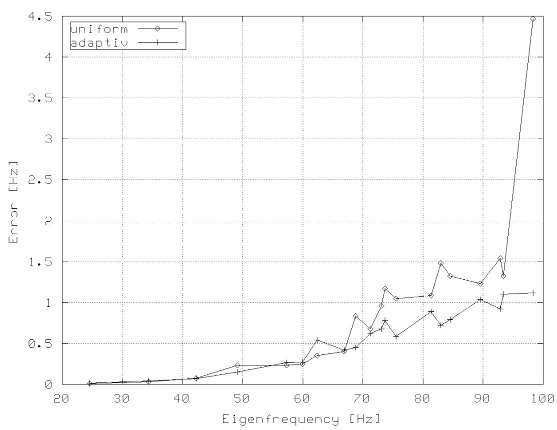

Figure 3: Eigenfrequency error

The uniform mesh was generated with 51329 nodes. The seed for the adaptive mesh was a coarse one, consisting of 1076 nodes. After six refinement steps, the adaptive mesh size was with 48454 nodes smaller than the one of the uniform mesh. Nevertheless the adaptive result beats the uniform one.

6 Conclusions

Using rules of thumb can be dangerous if one is not only interested in global field pressure errors. We demonstrated that one can find a significant discrepancy between local and global errors, even for the simple test case of a rectangular room.

Especially when solving a problem with a complex geometry, the usage of adaptive generated meshes is strongly recommended. In addition, hierarchical finite element spaces enable the usage of fast multigrid solvers with convergence rates, which are typically independent of the problem dimension.

Thus, adaptive multigrid makes sound field calculations fast and robust against discretisation problems.

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