

DISPLACEMENT VERSUS PRESSURE FORMULATIONS IN ELASTOACOUSTICS: FINITE ELEMENT APPROXIMATION

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Abstract: Two finite element methods for the numerical computation of fluid-structure vibration problems are considered. They are applied to two formulations based on different variables to describe the fluid: pressure in one case and displacements in the other. While in the first one the problem is discretized by standard Lagrangian finite elements for both variables, in the second one “face” Raviart-Thomas elements are used. In each case we consider tetrahedral meshes. The numerical results allow us to compare both methods in terms of error versus number of degrees of freedom and computing time.

1 Introduction

This paper is devoted to compare two different ways of solving the elastoacoustic problem by finite element methods. As it is well known, the elastoacoustic problem consists of the interaction of an acoustic fluid and an elastic solid. We are interested in harmonic vibrations and hence in modal analysis. Moreover we only consider bounded domains.

An overview of this problem and another fluid-structure interactions problems as well as their numerical solution can be found in the book by Morand and Ohayon[12].

In any mathematical model of the above mentioned problems, displacements are typically used to describe the solid. However several possibilities exist for the fluid. The first choice is to use pressure, as it is usually done in pure acoustics. The inconvenient is that, after discretization, we obtain non-symmetric generalized eigenvalue problems ([13]), whose computational solution involves considerable complications. To avoid this drawback the fluid has been described using alternative variables like velocity potential ([9]), displacement potential or combinations of some of them, like the pressure/potential formulation introduced in [11]. Only recently, the non-symmetric pressure formulation has been analyzed from a mathematical point of view in [4]. In this reference, it is also presented a numerical algorithm transforming the resulting non-symmetric matricial problem in another equivalent one, much more adequate for its numerical resolution.

Since the solid is generally described in terms of displacements, to choose the same variable for the fluid presents important advantages. Firstly, both kinematic and kinetic conditions at the fluid-structure interface are very easily handled. Secondly the method can be extended to nonlinear situations (see [1]). Finally it leads to sparse symmetric matrices. However, in principle, some disadvantages also arise: vector fields have to be computed instead of scalar fields and, moreover, Lagrangian finite elements produce spurious eigenvalues (see [10]). The latter can be avoided by using “face elements” (see [2, 6, 7]), which are somehow similar to the “edge elements” currently used in electromagnetism.

2 Mathematical Modelling

In this section we recall the equations for the harmonic vibrations of the coupling between a linear elastic structure and a compressible and inviscid fluid. The fluid is supposed to be barotropic, i.e., pressure only depends on density.

We consider the problem of determining the small amplitude motions of this fluid contained into a linear isotropic elastic structure which obeys Hooke's law. Let Ω_F and Ω_S be the domains occupied by fluid and solid, respectively. Let Γ_I be the interface between both media and $\vec{\nu}$ its unit normal vector pointing outwards Ω_F . The exterior boundary of the solid is the union of two parts, Γ_D and Γ_N : the structure is fixed on Γ_D and free on Γ_N . Finally let \vec{n} be the unit outward normal vector on Γ_N (see Figure 1).

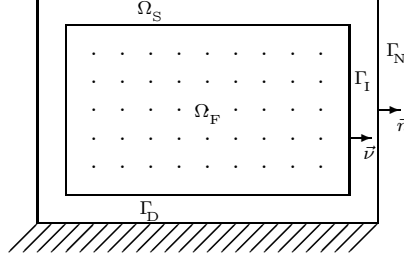


Figure 1: Elastoacoustic problem: vertical section of fluid and solid domains.

In the frequency domain, the governing equations for free small amplitude motions of the coupled system are the following:

$$-\omega^2 \rho_F \vec{u} + \text{grad } p = \vec{0} \quad \text{in } \Omega_F, \quad (2.1)$$

$$\omega^2 \rho_S \vec{w} + \text{div}(\boldsymbol{\sigma}(\vec{w})) = \vec{0} \quad \text{in } \Omega_S, \quad (2.2)$$

$$p + \rho_F c^2 \text{div } \vec{u} = 0 \quad \text{in } \Omega_F, \quad (2.3)$$

$$(\vec{u} - \vec{w}) \cdot \vec{\nu} = 0 \quad \text{on } \Gamma_I, \quad (2.4)$$

$$p \vec{\nu} + \boldsymbol{\sigma}(\vec{w}) \vec{\nu} = \vec{0} \quad \text{on } \Gamma_I, \quad (2.5)$$

$$\vec{w} = \vec{0} \quad \text{on } \Gamma_D, \quad (2.6)$$

$$\boldsymbol{\sigma}(\vec{w}) \vec{n} = \vec{0} \quad \text{on } \Gamma_N, \quad (2.7)$$

where p is the amplitude of the fluid pressure, \vec{u} and \vec{w} are those of fluid and solid displacements, ω is the angular frequency, ρ_F and ρ_S the fluid and solid densities, c is the sound speed in the fluid, and $\boldsymbol{\sigma}$ is the stress tensor in the solid, which is related to the linearized strain tensor $\boldsymbol{\varepsilon}(\vec{w}) := \frac{1}{2}(\nabla \vec{w} + \nabla \vec{w}^t)$ by Hooke's law: $\boldsymbol{\sigma} = \lambda(\text{tr } \boldsymbol{\varepsilon}(\vec{w}))\mathbf{I} + \mu \boldsymbol{\varepsilon}(\vec{w})$ (λ and μ being the Lamé coefficients).

3 Weak formulations

In this section we recall two different weak formulations for the above model. They are obtained by eliminating some of the unknowns describing the fluid.

3.1 Pressure formulation

It was first considered in reference [13] (see also [14] and [11]) and can be obtained by eliminating the fluid displacement field from the above equations. Then the following weak formulation is attained by standard procedures, i.e., multiplying by virtual displacement and pressure fields and then integrating by parts using a Green's formula:

Find $\omega \in \mathbb{R}$, $\vec{w} \in H_{\Gamma_D}^1(\Omega_S)^3$ and $P \in H^1(\Omega_F)$, with $(\vec{w}, P) \neq 0$, such that

$$\begin{aligned} & \int_{\Omega_S} \boldsymbol{\sigma}(\vec{w}) : \boldsymbol{\varepsilon}(\vec{z}) \, dx + \frac{1}{\rho_F} \int_{\Omega_F} \text{grad } p \cdot \text{grad } q \, dx - \int_{\Gamma_I} p \vec{z} \cdot \vec{\nu} \, d\Gamma \\ & = \omega^2 \left(\int_{\Omega_S} \rho_S \vec{w} \cdot \vec{z} \, dx + \frac{1}{\rho_F c^2} \int_{\Omega_F} p q \, dx + \int_{\Gamma_I} \vec{w} \cdot \vec{\nu} \, q \, d\Gamma \right) \\ & \quad \forall \vec{z} \in H_{\Gamma_D}^1(\Omega_S)^3, \quad \forall q \in H^1(\Omega_F). \end{aligned}$$

In the expression above, $H_{\Gamma_D}^1(\Omega_S)$ is the subspace of functions square integrable and which derivatives also square integrables, vanishing on Γ_D .

3.2 Displacement formulation

It is obtained by using equation (2.3) to eliminate pressure in equations (2.1) and (2.5). Consider the space of displacements in each media satisfying the kinematic constraint (2.4):

$$\mathcal{V} := \left\{ (\vec{u}, \vec{w}) \in H(\text{div}, \Omega_F) \times H_{\Gamma_D}^1(\Omega_S)^3 : \vec{u} \cdot \vec{\nu} = \vec{w} \cdot \vec{\nu} \text{ on } \Gamma_I \right\},$$

where $H(\text{div}, \Omega_F)$ denotes the space of square integrable vector fields the divergence of which is also square integrable. The weak formulation of the displacement formulation looks very simple and compact. Moreover it is symmetric:

Find $\omega \in \mathbb{R}$ and $(\vec{u}, \vec{w}) \in \mathcal{V}$, with $(\vec{u}, \vec{w}) \neq 0$, such that

$$\int_{\Omega_S} \boldsymbol{\sigma}(\vec{w}) : \boldsymbol{\varepsilon}(\vec{z}) dx + \int_{\Omega_F} \rho_F c^2 \text{div } \vec{u} \text{div } \vec{y} dx = \omega^2 \left(\int_{\Omega_S} \rho_S \vec{w} \cdot \vec{z} dx + \int_{\Omega_F} \rho_F \vec{u} \cdot \vec{y} dx \right), \quad \forall (\vec{y}, \vec{z}) \in \mathcal{V}.$$

This formulation has been discretized by standard nodal finite elements in [10]. However, in such case, spurious eigenvalues arise among the physical ones. An alternative approach has been proposed in [7] and analyzed in [2] and [3]. It consists of using Raviart-Thomas finite elements, which are $H(\text{div})$ -conforming, to discretize the fluid displacements. The kinematic interface condition can be taken into account either in a weak sense by using a Lagrange multiplier or by eliminating the degrees of freedom of the fluid on the interface by static condensation. Non-compatible meshes on the fluid-solid interface can be used which is very convenient to deal with singularities arising from reentrant corners or dihedral angles in the solid (see [6] and [5]).

4 Discretization

In this section we introduce finite element discretizations for solving the two formulations in the previous section. In both cases we consider tetrahedral meshes, but the methods can be directly applied to hexahedral meshes (see [5]).

4.1 Pressure formulation

We discretize the solid displacements using standard piecewise linear and continuous elements on each variable. To discretize the fluid pressure we use the same standard Lagrangian elements. If W and P denote the column vectors of components of \vec{w} and p , respectively, in the standard finite element basis, the discrete pression formulation can be written in matricial form as follows:

$$\begin{pmatrix} K_S & -A \\ 0 & \frac{1}{\rho_F} F \end{pmatrix} \begin{pmatrix} W \\ P \end{pmatrix} = \omega_h^2 \begin{pmatrix} M_S & 0 \\ A^t & K_P \end{pmatrix} \begin{pmatrix} W \\ P \end{pmatrix},$$

where

$$\begin{aligned} \int_{\Omega_S} \boldsymbol{\sigma}(\vec{w}) : \boldsymbol{\varepsilon}(\vec{z}) dx &= Z^t K_S W, & \int_{\Omega_S} \rho_S \vec{w} \cdot \vec{z} dx &= Z^t M_S W, \\ \int_{\Omega_F} \vec{\text{grad}} p \cdot \vec{\text{grad}} q dx &= Q^t F P, & \frac{1}{\rho_F c^2} \int_{\Omega_F} p q dx &= Q^t K_P P, \\ \int_{\Gamma_I} p \vec{z} \cdot \vec{\nu} d\Gamma &= Z^t A P. \end{aligned}$$

We can apply elementary transformations to have an equivalent system with the matrix in the right hand side symmetric and positive definite,

$$\begin{pmatrix} K_S & -A \\ 0 & \delta \frac{1}{\rho_F} F \end{pmatrix} \begin{pmatrix} W \\ P \end{pmatrix} = \frac{\omega_h^2}{1 - \delta \omega_h^2} \begin{pmatrix} M_S - \delta K_S & \delta A \\ \delta A^t & \delta \left(K_P - \delta \frac{1}{\rho_F} F \right) \end{pmatrix} \begin{pmatrix} W \\ P \end{pmatrix},$$

where δ depends on the solution of uncoupled spectral problems for the fluid and the solid (see [4] for details). Notice that the matrix on the left-hand side is not symmetric.

4.2 Displacement formulation

We consider the same standard piecewise linear and continuous elements to discretize each variable on the solid displacement. For the fluid displacement we choose the lowest degree Raviart-Thomas tetrahedral finite elements. In this case the fluid displacements are approximated by incomplete polynomials of degree one in each tetrahedron and the degrees of freedom are the (constant) normal components on each face. The interface condition is taken in a weak sense. By so doing, spurious modes do not appear (see [2, 6, 7]). If W and U denote the column vectors of components of \vec{w} and \vec{u} , respectively, in the respective finite element basis, the matricial formulation for the approximated problem can be written

$$\begin{pmatrix} K_S & 0 \\ 0 & K_F \end{pmatrix} \begin{pmatrix} W \\ U \end{pmatrix} = \omega_h^2 \begin{pmatrix} M_S & 0 \\ 0 & M_F \end{pmatrix} \begin{pmatrix} W \\ U \end{pmatrix},$$

where K_S and M_S have been already defined and

$$\int_{\Omega_F} \rho_F c^2 \operatorname{div} \vec{u} \operatorname{div} \vec{y} dx = Y^t K_F U, \quad \int_{\Omega_F} \rho_F \vec{u} \cdot \vec{y} dx = Y^t M_F U.$$

However, to solve the coupled problem, we need to impose the kinematic constraint (2.4) in a weak sense. If the fluid nodal components corresponding to the faces lying on the interface are statically condensed in terms of the nodal values of the solid displacements (see [5]), we obtain the following symmetric eigenvalue problem

$$\begin{pmatrix} \widehat{K}_F & C_K^t E \\ E^t C_K & K_S + E^t K_F^1 E \end{pmatrix} \begin{pmatrix} W \\ \widehat{U} \end{pmatrix} = \omega_h^2 \begin{pmatrix} \widehat{M}_F & C_M^t E \\ E^t C_M & M_S + E^t M_F^1 E \end{pmatrix} \begin{pmatrix} W \\ \widehat{U} \end{pmatrix},$$

where \widehat{U} is the column vector corresponding to the remaining fluid nodal components. We notice that both matrices are symmetric and the one on the right-hand side is positive definite.

5 Numerical Results

In this section we consider a test problem corresponding to water in a perfectly rigid cavity covered by a 3D clamped moderately thick steel plate (see Figure 2). The dimensions for the cavity are 4 m \times 6 m \times 1 m and for the plate 4 m \times 6 m \times 0.5 m. The physical data are $\rho_F = 1000 \text{ kg/m}^3$ and $c = 1430 \text{ m/s}$ for water, and $\rho_s = 7700 \text{ kg/m}^3$, Young modulus $E = 1.44 \times 10^{11} \text{ Pa}$ and Poisson ratio $\nu = 0.35$ for steel. We have solved the problem in a quarter of the geometry to reduce the number of degrees of freedom, by imposing the symmetries of the different modes as constraints. As reference solution we have used that obtained by using a plate model (Reissner-Mindlin) for a fine mesh coupled with Raviart-Thomas elements for the fluid (see [8]).

Figures 4 to 7 show the error versus either the number N of degrees of freedom (d.o.f) or the computer time required to calculate the four eigenvalues nearest to a “shift” parameter by using Version 5.3 of MATLAB eigensolver `eigs`. This eigensolver computes, by means of Arnoldi iterations, the eigenvalues and eigenvectors of a generalized eigenvalue problem of the form $\mathcal{A}x = \lambda \mathcal{B}x$ with \mathcal{B} symmetric and positive definite.

We observe from these figures that the pressure formulation is slightly better than the displacement formulation except for coarse meshes.

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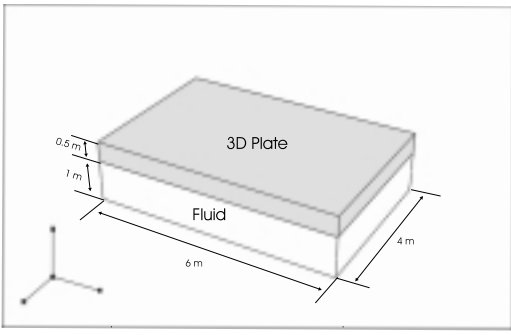


Figure 2: Rigid 3D cavity covered by a thick plate.

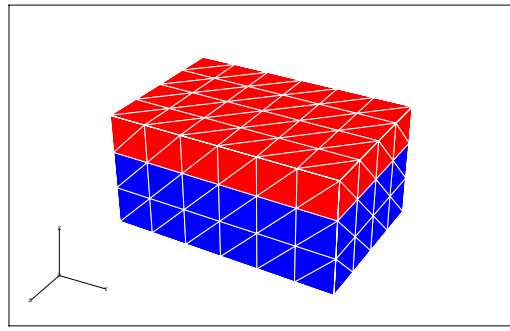


Figure 3: Tetrahedral mesh (quarter of the geometry).

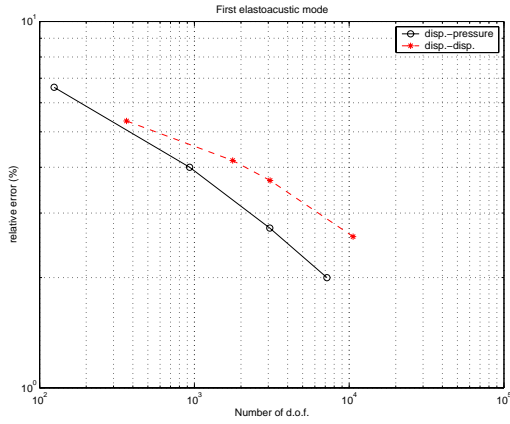


Figure 4: Error versus number of d.o.f (log-log scale).

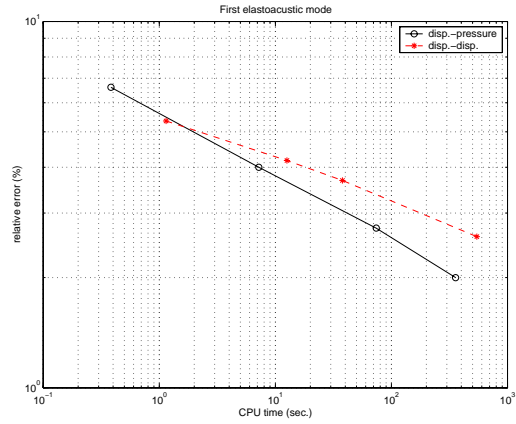


Figure 5: Error versus computer time (log-log scale).

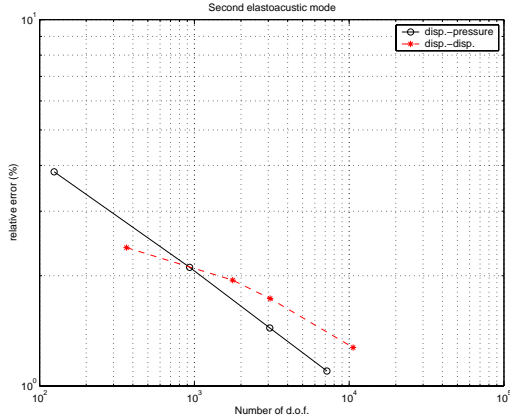


Figure 6: Error versus number of d.o.f (log-log scale).

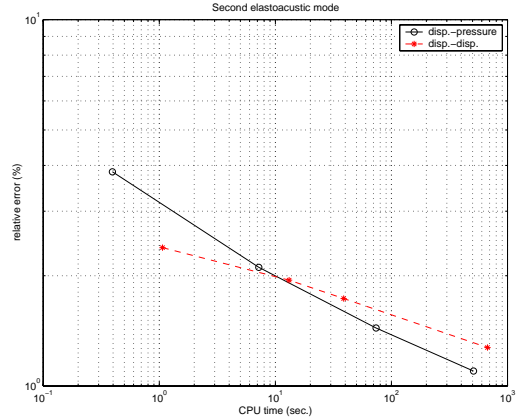


Figure 7: Error versus computer time (log-log scale).

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