ON THE FREE VIBRATION OF PLATES

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Nieves, F.J.[1]; Bayón, A.[2]; Gascón, F.[3]
1) ETS de Arquitectura, Universidad de Sevilla, Reina Mercedes, 2, 41012 Sevilla, Spain.
Tel 34 954556627, Fax 34 954557892, E-mail nieves@us.es
2) ETSI de Minas, Universidad Politécnica de Madrid, Ríos Rosas, 21, 28003 Madrid, Spain,
Tel 34 913367072, Fax 34 913366952, E-mail abayon@dfarn.upm.es
3) ETS de Arquitectura, Universidad de Sevilla, Reina Mercedes, 2, 41012 Sevilla, Spain.
Tel 34 954556612, Fax 34 954557892, E-mail fgascon@us.es

ABSTRACT

The out-of-plane displacement component of an aluminium plate-shaped rectangular parallelepiped, vibrating freely, is experimentally detected by a laser interferometer. The vibration spectrum is determined by means of the Fourier transform, which gives the lowest natural frequencies of the parallelepiped. In this study, flexural symmetric modes are preferably analysed. The natural frequencies of the plate are firstly calculated by applying the one-dimensional beam theory and the two-dimensional plate theory of linear elastic plates. Then, the Ritz method with products of powers of the co-ordinates as basis functions is applied to obtain the lowest flexural natural frequencies. Three-dimensional solutions are obtained, unlike those provided by simpler theories. The experimental results are compared with the theoretical predictions.

INTRODUCTION

The vibration of the elastic plates has been a subject widely treated, both from experimental and theoretical points of view[1], which is due to its multiple practical applications. The vibration spectrum of a plate can be excited and detected with an appropriate experimental set-up. A correct interpretation of the results allows us to obtain useful information. Some works are focused on free-vibration experiments[2], others refer to plates vibrating under external excitation[3]. Transducers[3] or optical interferometers[4] are usually used as detectors.

The elementary theory of linear elastic beams is based on the principle of Saint Venant, according to which, far from the points of application of forces, the behavior of a slender beam depends only on the resultant force and the resultant moment on each section. The elastic potential of a beam depends mainly on the bending moment, while tensile and shear stresses are usually neglected when nonnull bending moment exists. The plate is the two-dimensional analogue of the beam. The elementary theory of plates is based on the theory of beams; the only internal force that appears explicitly on that theory is the bending moment. The displacement of the neutral surface of a plate is usually assumed to be perpendicular to the plate. With this approach, the displacement is unidirectional and depends on the two coordinates of the points of the plate. This method provides the analytical solution of certain dynamic problems.

The variational methods are a powerful tool of calculation. One of the used procedures is the Ritz method. His inventor already applied it to the study of the free vibration of a plate a century ago. As it is well known, the Ritz method proposes an suitable set of basis functions, depending on the coordinates. The displacements are assumed to be a sum of such functions multiplied by constants[3,5]. We will use as basis functions power series in the co-ordinates to certain powers. This selection is correct from the mathematical point of view as well as suitable from the conceptual point of view.

In this work we have generated and detected an almost free vibration of an elastic test sample. The results are compared with those predicted by some elementary analytical theories and those calculated using the Ritz method implemented in Maple.

EXPERIMENTAL SET-UP AND RESULTS

An aluminium plate, rectangular parallelepiped-shaped, with dimensions L_1= 0.09990 m, L_2= 0.15100 m, and L_3= 0.02490 m, was used for the tests in the laboratory. The mass m of the sample is 0.9951 kg, then its density ρ =2649 kg/m^3. Fig. 1 shows the plate and the set of co-ordinate axes with its origin coincident with the centre of the plate. The *P-* and *S-waves* velocities have been measured in the aluminium plate by the pulse velocity method. The transit times for a path length of 2×0.02490 m were t_p=8.030×10^{-6} s and t_s=15.588×10^{-6} s for the *P*-wave and *S*-wave, respectively. Consequently, the velocities are equal to c_p=6202 m/s and c_s=3195 m/s. The elastic constants can be determined from the very well-known relations:

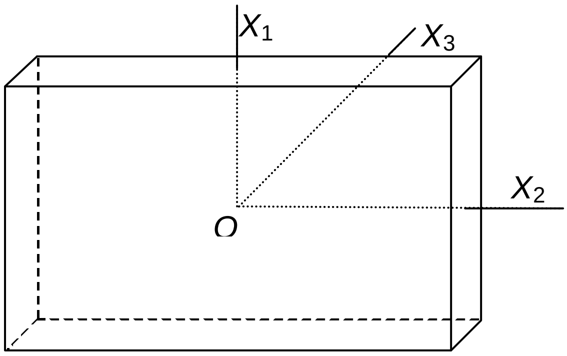

Fig.1

-Shear modulus

$$G = \rho c_s^2 = \frac{4mL_3}{L_1 L_2 t_s^2} \tag{1}$$

-Poisson's ratio

$$\nu = \frac{(c_p/c_s)^2 - 2}{2(c_p/c_s)^2 - 2} = \frac{(t_s/t_p)^2 - 2}{2(t_s/t_p)^2 - 2} \tag{2}$$

The magnitudes measured directly in the laboratory appear explicitly in the third term of (1) and (2). The values of the elastic constants obtained for the aluminium plate are G=27.04 GPa and ν=0.3194 (Young's modulus being E=71.35 GPa).

The experimental set-up for quasi free vibration is shown in Fig. 2. The plate with its face x_3=L_3/2 forward the detection system is supported on two small rubber blocks. Therefore, its movement is softly restrained. Symmetrical flexural vibration is induced by applying at the central point on x_3= -L_3/2 an impact perpendicular to the plate. A pendulum consisting of a thread and a steel sphere was used to strike the sample. This type of excitation allows the plate to oscillate freely in its natural flexural modes, since following the impact no additional forces act upon the sample.

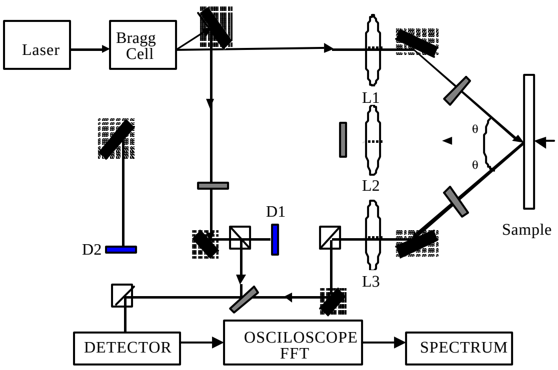

Fig. 2

The resulting vibration was detected at the central point on x_3=L_3/2. The detection system was a laser interferometer OP-35 I/O (Ultra Optec Inc.). With this system, out-of-plane or in-plane displacement components can be detected at a point with a resolution in amplitude of about 1 nm. In our case, for flexural oscillations, the measured component of the displacements was the out-of-plane. The detection principle is based on the speckle phenomenon, which is observed when coherent light strikes on a scattering surface producing a pattern with bright and dark spots. The size of the illuminated area is approximately 20 µm; consequently, detection is point-like and without contact. The bandwidth ranges from 1 kHz to 35 MHz, allowing simultaneous detection of several vibration modes.

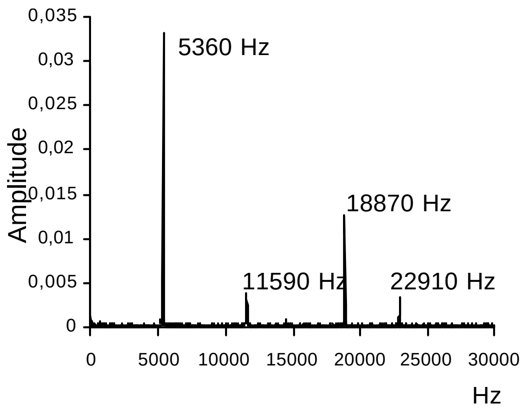

Fig. 3

The interferometer works in the out-of-plane configuration as it is shown in Fig. 2. The laser beam is splitted in two by a Bragg cell; one with the same frequency as the original, the other with a frequency shifted by 40 MHz. The undeflected beam is focused on the surface of the sample. The resulting scattered light is collected in the direction symmetrical to that of the incident beam and then led to the beam mixer where interferes with the reference beam. An out-of-plane displacement δw, in the OX_3-direction, causes a phase change equal to $4\pi\cos\theta\,\delta w/\lambda$, where λ is the wavelength of the laser. A 40-MHz frequency signal, modulated in phase by the displacements, is obtained in the detector. The signal is processed by an demodulating unit to yield an output proportional to instantaneous displacement of the surface at the detection point. Finally, a Tektronic TDS-430A oscilloscope digitises the signal and the spectrum of the vibration is calculated using the fast Fourier transform (FFT). The natural frequencies will be those associated with the maximum amplitudes in the spectrum. The spectrum obtained for our sample is that of Fig. 3. The lowest flexural frequencies detected are f_1=5360 Hz, f_2=11590 Hz, and f_3=18870 Hz.

UNCERTAINTIES

Since one of the objectives of this paper is to compare the experimental results with the theoretical ones, it seems convenient to calculate the uncertainty of the experimental measures of directly or indirectly obtained magnitudes.

Let us apply the systematic uncertainty methodology. As it is known[6], if a physical magnitude y is a function of a set of physical magnitudes x_i, $y=F(\{x_i\})$, which have been measured directly, and they are affected by their respective uncertainties U_{x_i}, then the uncertainty of an indirect measurement U_y is estimated by means of the differential of this function using the absolute values of the partial derivatives, that is, $U_y = \Sigma\left|\partial F/\partial x_i\right| U_{x_i}$. We will suppose that all the measuring instruments are well calibrated, then their U_{x_i} uncertainties are only due to their sensitivities.

The resolution of the used Fourier analyzer is 10 Hz. The uncertainties in the measurement of lengths, mass and transit times are $5*10^{-5}$ m, 10^{-4} kg and 10^{-9} s, respectively. Then, the absolute value of the systematic uncertainties in the indirect measurement of the shear modulus, deduced from (1), is U_G=0.08 Gpa. Analogously from (2) it is found for the Poisson's ratio U_ν=0.0001. The values obtained for the uncertainties of G and ν must be considered minimum values, because random uncertainties or calibration errors have not been taken into account.

ELEMENTARY THEORIES

As the problem of the general analytical calculation of the vibration mode shapes and natural frequencies of plates is too complicated, the obtained solutions are limited to particular cases.

The used sample is longer than high and higher than wide, so it can be considered in a first approach like a rod of length L_2, area of transversal section $A=L_1L_3$ and moment of inertia $I=AL_3^2/12$. The equation for the frequencies is deduced from the theory of flexural free vibration of finite beams with simple free-free boundary condition:[7]

$$f_n = \frac{(\beta_n L_2)^2}{2\pi L_2^2}\sqrt{\frac{EI}{\rho A}} \,. \qquad (3)$$

Since the first non-zero mode corresponds to $\beta_1 L_2 = 4.730$, it results for our plate f=5826 Hz.

A second possibility of calculation is inferred from books of elasticity[8] for thin plates. It is found that the velocity for flexural waves, when displacements are considered out-of-plane, $u_1=u_2=0$, $u_3= u_3(x_1,x_2,t)$, is given by

$$c = \frac{2\pi L_3}{\lambda}\sqrt{\frac{E}{3\rho(1-\nu^2)}} \,. \qquad (4)$$

When a plate vibrates with free boundary conditions, an expectable mode is the simplest one of symmetrical bending, where the plate presents a longitudinal section seemed to the curve s_1 drawn in Fig. 4. Since a plate is easily deformable in the shown shape, very low frequencies will have to correspond to them. It is assumed in the elementary theory, that the first symmetrical mode can be represented by a period of a function cosine, reason why the wavelength equals to the length L_2 of the plate. With these assumptions, the smaller frequency results to be

$$f = \frac{2\pi L_3}{L_2^2}\sqrt{\frac{E}{3\rho(1-\nu^2)}} \qquad (5)$$

For the plate available in the laboratory, the mode with lower frequency has a frequency of f=21696 Hz. The expectable modes s_1, a_1 and s_2 have been drawn in Fig. 4, in principle in agreement with the elementary theory. Note that all the modes have a maximum of amplitude in the edges and that the symmetrical ones have another maximum in the center. Since we are specially interested in the symmetrical modes, we should excite them by applying an impact on the plate in the center. In addition, as the displacements are also detected in the center, we exclude the antisymmetrical vibration modes possibly generated by lack of precision in the impact. These reasons have led us to impact and detect in the center. Correct mode shapes are not known a priori, but it will be always truth that the symmetrical ones will have a maximum or antinode in the center and antisymmetrical ones a null value or node at this point, therefore the percussion must be applied into the center and the detection must be carried out in the center too.

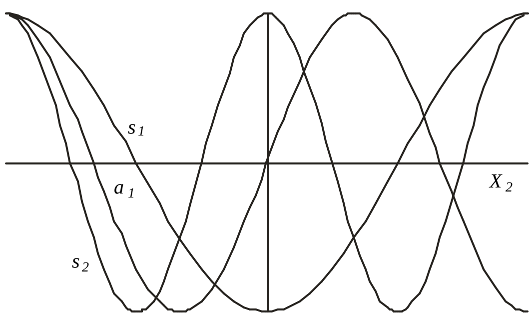

Fig. 4

THE RITZ METHOD

As it is well known, the Hamilton's principle postulates as fundamental equation of dynamics for a free system that the integral between two given instants of the Lagrange's functional of a system of particles is stationary. For a harmonic solution of the type $u_i=U_i(x_1,x_2,x_3)\sin(\omega t+\phi)$, it is enough to consider the maximum kinetic energy in a period of the motion

$$T_m = \frac{1}{2}\omega^2\int_\tau \rho(U_1^2+U_2^2+U_3^2)\,d\tau \qquad (6)$$

and the maximum potential energy

$$V_m = G\int_\tau \left(\varepsilon_{ij}\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ii}\varepsilon_{ii}\right)d\tau \qquad (7)$$

where ε_{ij} are calculated from U_k.

In the method of Ritz, a solution for the displacements is proposed like a linear combination of an suitable set of basis functions[9] that verify the boundary conditions for the displacements, if these are predetermined, which is not our case. Adequate functions, chosen by us, are monomials formed by products of powers of the coordinates. The formed algebraic polynomials have unknown coefficients, whose values are found out by minimizing the difference of the maximum kinetic and potential energies, that is, the partial derivatives of such a difference respect to each coefficient must be null. The natural frequencies of vibration are obtained from the condition of compatibility of the set of equations, and the solution of the system gives the eigenvectors or coefficients. The method has as advantage that the solution can be obtained with the desired precision, except for the limitations of the computer and the spent time of computation. In addition, the obtained frequencies are always higher than the corresponding ones to the correct solution, reason why it is not difficult to find out the convergence by adding simply a new term to the polynomial. As a test of simplicity of the procedure, we have proposed a solution as simple as $U_1=U_2=0$, $U_3=a+bx_2^2$, which must correspond to a mode of symmetrical flexion. A manual calculation, that occupies less than an A4-size page, leads to the solution

$$w^2 = \frac{60G}{rL_2^2} \tag{8}$$

When applying this result to our plate, its frequency results to be f=26094 Hz.

We have developed a Maple program to apply the Ritz method to the study of the free vibrations of a plate. The abovementioned monomials were taken as the basis functions. In order to verify its accuracy, we have proposed a problem of vibrations of a cube which has already been solved by the Ritz method with Legendre polynomials as well as verified experimentally[3]. It is found a great agreement for the frequencies calculated by us for the lowest modes, which verified the goodness of our program. The used computer has been a PC, Pentium II.

In a second approach for our plate, we propose a solution similar to that used in the two-dimensional theory of plates[2,5], in which $U_1= U_2= 0$, $U_3= U_3(x_1, x_2)$. Therefore it is a bidimensional theory. This time we have used for U_3 a polynomial of degree $P+Q$ such as

$$U_3 = \sum_{p=0,q=0}^{P,Q} A_{3pq} x_1^{\,p} x_2^{\,q} \, . \tag{9}$$

If the study is limited to the flexural symmetrical modes, the exponents of all the coordinates must be even. In the first calculation the exponents p,q have reached the maximum values P=0 and Q=2; with these values the lowest natural frequecy is 26084.3 Hz. When repeating the calculation for 2, 0, we have got 39426.8 Hz for such frequency. Thus the first natural frequency is smaller for the first trial. As the smallest value of the frequency is always the nearest to the exact solution, we deduce that the dependency of U_3 is stronger on the coordinate x_2 and weaker on the coordinate x_1, as expected by applying the elementary theory. Following this observation, the exponents have been increased and the calculated three first lowest natural frequencies for the exponents 8, 8, are 21158.5, 31981.2 and 38346.9 Hz. These results obtained for P and Q equal to 8 have been reached in a smaller time than a minute and the six significant digits are the same as those calculated with values for the powers as high as 14, 14, which means 64 unknown coefficients.

In a third approach, instead of using the bidimensional theory of plates, the general case is studied. Displacements are assumed to depend on the three spatial coordinates and the time (three-dimensional theory). The study is focused on the lowest symmetrical modes. The mode shapes are expected to be similar to those represented in Fig. 4. In order to compare the results of the diverse starting hypotheses, we have proposed a first trial of components of the displacement such as $U_1=A_{1101}x_1x_3$, $U_2=A_{2011}x_2x_3$ and the third component has been of the type (9). With this hypothesis, the values of frequencies obtained were 6231.0, 13770.8 and 38346.9 Hz. These values represent an important quantitative jump favourable to the 3D-hypothesis.

In general and considering the symmetries of the problem in the solution, a totally three-dimensional solution is given by:

$$U_1 = \sum_{p,q,r}^{P_1,Q_1,R_1} A_{1pqr} x_1^{\,p} x_2^{\,q} x_3^{\,r} \tag{10}$$

$$U_2 = \sum_{p,q,r}^{P_2,Q_2,R_2} A_{2pqr} x_1^p x_2^q x_3^r \quad (11)$$

$$U_3 = \sum_{p,q,r}^{P_3,Q_3,R_3} A_{3pqr} x_1^p x_2^q x_3^r \quad (12)$$

where the exponents of the coordinates in the formula of U_1 are successively odd, even, odd, while in the formula of U_2 are even, odd, odd and in the one of U_3 they are even, even, even.

With the smallest possible values for P_1, Q_1, …, R_3, that is 1, 2, 1; 2, 1, 1; 2, 2, 2, respectively and after a computation time of half a minute, the obtained frequencies were 6018.6 Hz, 13104.3 Hz and 24091.6 Hz. Insisting on the calculation and after five hours of computation, the three lower frequencies 5345, 11602, and 19050 Hz were obtained, with the exponents up to 7, 6, 7; 6, 7, 7; 6, 6, 6, which means 192 unknown coefficients.

The systematic uncertainty in the numerical calculation of the frequencies was estimated by repeating the last calculation but adding, or substracting if it was required, to the lengths of the sample their corresponding uncertainties. The same process was repeated for the values of m, t_s and t_p. This uncertainty has turned out to be 28 Hz, 62 Hz, 91 Hz respectively for the three lowest frequencies. Note the coincidence of the experimental result 5360±10 Hz and the one of the 3D-theory, 5345 ± 28 Hz.

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