Wave propagation in self-similar structures

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ABSTRACT : As for Cantor-like structures, structures generated from the Sierpinski carpet present remarkable vibrational behaviours such as localisation phenomena and scale effects associated with their self-similar geometry.
The singular characteristics of structures based on the Cantor set have been discussed since 1992 by Petri et al. [1,2] in ultrasonics and Gibiat et al. [3] in audible range. In particular, it has been shown that two types of vibrational modes can exist in such structures : extended modes (phonons) where the energy is distributed all along the structure, and localised modes (fractons) where energy is trapped in just a fraction of the structure. The frequencies corresponding to these type of modes can be predicted recursively from those of lower order structures. In this paper we will present similar behaviours that we have observed in structures whose geometry has been generated following that of the Sierpinski carpet.

I –INTRODUCTION

In a former work [Gibiat, 2002] a theoretical and experimental study of the acoustical propagation in a waveguide with a Cantor-like structure was presented. One of the main results is that of the existence of trapped modes and scale effects related with the degree of self-similarity, results that are in good agreement with the previous results obtained by Petri et al. in ultrasonics [Petri 1992]. The present works deals with the same kind of study on a mechanical 2D system consisting of a square membrane loaded with masses (positive or negative -i.e. holes-) whose positions are chosen so that the final object shows a geometry close to that of the Sierpinski carpet [Mandelbrot, 1982] .
The loaded membrane has been studied through numerical simulations based on an algorithm following the same philosophy that the cellular automaton presented in Barjau et al. [Barjau 2002].

In the first section we will present the iterative construction of the pre-fractal membrane. The second one will be devoted to the basic concepts leading to the numerical algorithm that has been used. The last section will give some results showing that trapped modes are present on this membrane and that the same kind of scale effects are detectable as in the Cantor-like system.

II – ITERATIVE CONSTRUCTION OF THE SIERPINSKY MEMBRANE

The iterative process to build a Sierpinski membrane is the following: from a square uniform surface (which will be called Sierpinski order 0), the surface is divided into nine equal squares. The corners of the central one are then loaded with positive or negative masses (Sierpinski order 1). This operation (division and loading) is repeated for the eight remaining squares, thus generating a Sierpinski order 2. This process is then iterated giving the higher order structures (figure 1). It is obvious that this building process cannot be iterated up to infinity (real fractal state) for practical reasons. This is why we will speak in what follows of pre-fractal objects. As it is not possible to load the membrane with masses infinitely small, the frequency range of our study is (in terms of wavelength and so of frequency) limited by the size of the masses. Anyway we know from the results obtained on the Cantor-like Waveguide that the most important physical behaviours appear already in low order quasi-fractal structures.

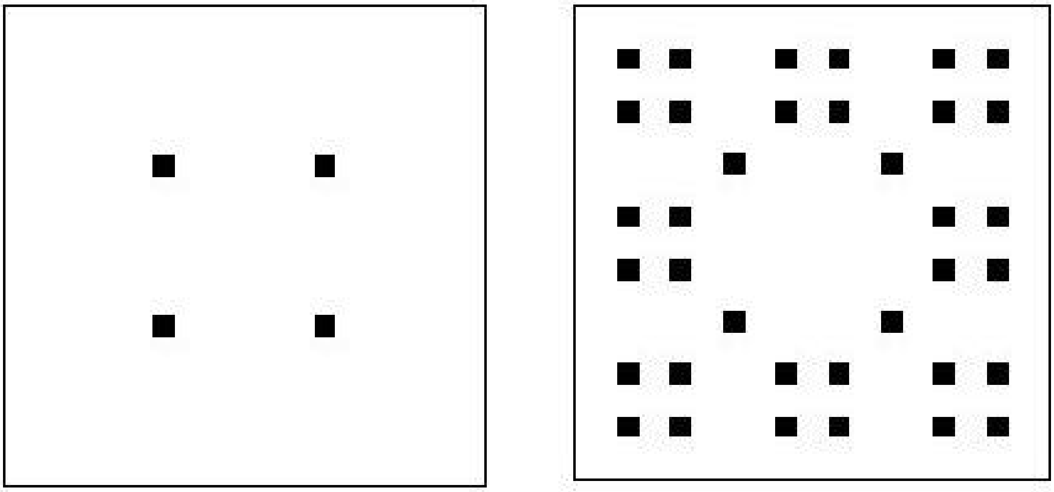


Figure 1 : Order 1 and 2 of the structure

III – WAVE PROPAGATION ON THE PREFRACTAL MEMBRANES

The propagation of transverse waves on such a membrane can be studied through different approaches. A classical analytical approach is possible through a perturbation method The frequencies and the shapes of the modes of the loaded membrane are expressed as a superposition of the modal eigenfunctions of the conservative uniform membrane. Actually, the modal basis is truncated, thus keeping a finite number of classical modes. The truncation criterium is chosen according to the size of the added masses (as said in the previous section).

A second approach is that of a finite difference approach. It is based on the discretisation of the wave equation governing the wave propagation on the membrane. The main problems in this classic method are the degree of discretisation of the surface, the description of the defects and the huge amount of memory needed for the convergence of the solution.

We have used a third and different approach. The basic idea consists in replacing the problem of the continuous membrane loaded by masses, by a discrete problem of a mesh of punctual masses (some corresponding to the membrane and some to the added masses) connected by springs. The geometry of the mesh can be chosen with different criteria. The two usual geometries are the square one or the hexagonal one (figure 2).

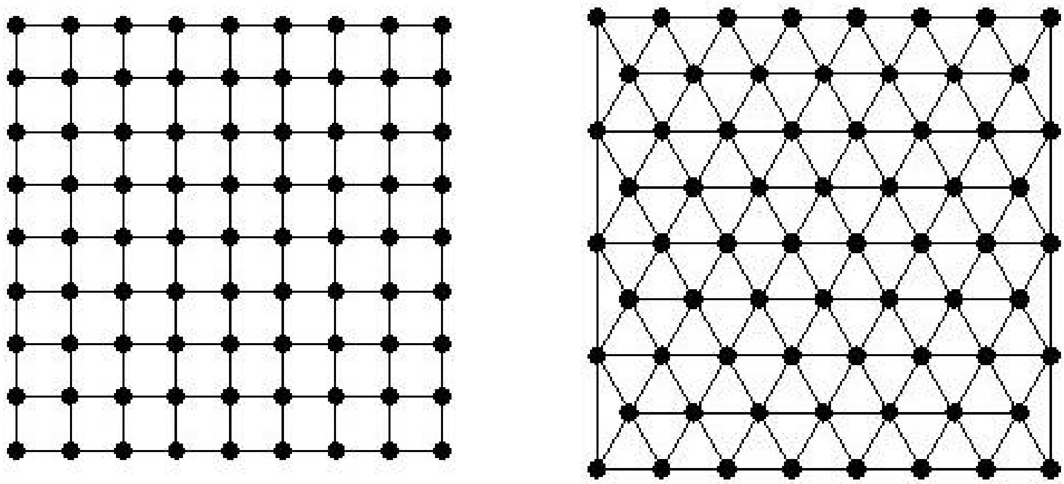


Figure 2: different mesh geometries.

The real physical process consisting on a propagation of kinetic and elastic energy, is simplified so that just one kind of energy is taken into account (somehow this can be seen as a sampled solution where only the states of maximum velocity or maximum deformation are retained). It is then possible to establish a simple algorithm describing this elastic or kinetic process in a similar way as done by Barjau et al. [Barjau, 2002] for the case of a 1D system. For the case of a square mesh, this algorithm coincides with that of Smith et al. [Smith, 1992], even if the philosophy of construction is totally different. This approach has proved to be very efficient and is less power consuming than the classical finite difference methods.

As for the 1D systems studied by Barjau et al. [Barjau, 2002], the algorithm for the 2D membrane coincides with that governing the acoustical propagation in mesh of uniform cylindrical connected pipes (each connection or node would correspond to a punctual mass). The basic rules leading to the elastic cellullar automaton algorithm (ECA) are the Kirchoff conditions at each node. Figure 3 shows their formulation for the case of a 2,3 and 4-branches node. The application of these rules is known as “collision phase”. Each “collision phase” is followed by a “propagation phase” where the new values of the variables are shifted to the neighbour nodes.

The implementation of the propagation phase shows the main difference implied in the use of square and hexagonal geometries. Figure 4 shows the evolution of the wave front in a uniform membrane associated with a single point perturbation for both geometries. The black dots represent the new points reached by the perturbation at each time step Δt. If the fundamental length of the mesh Δl is related to the time step through the wave propagation speed c, $\Delta l = c\Delta t$, it is clear form this figure that the square mesh implies a wave propagation speed of $c/\sqrt{2}$ along the diagonals. As a consequence, the wave front will not be circular as expected, and the wave intensity will not be uniform. This problem is encountered in finite difference schemes too.

As a first approximation, we have implement the square ECA. Future work will deal with the implementation of the hexagonal algorithm. As the computation occurs only on the connected points the complexity of the computation is small enough to be done on small computers.

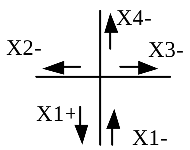
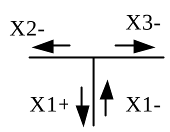
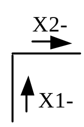
		
RN = -1/2 TN = 1/2	RN = -1/3 TN = 2/3	RN = 0 TN = 1

Figure 3 the cases 4 branches 3 branches and 2 branches

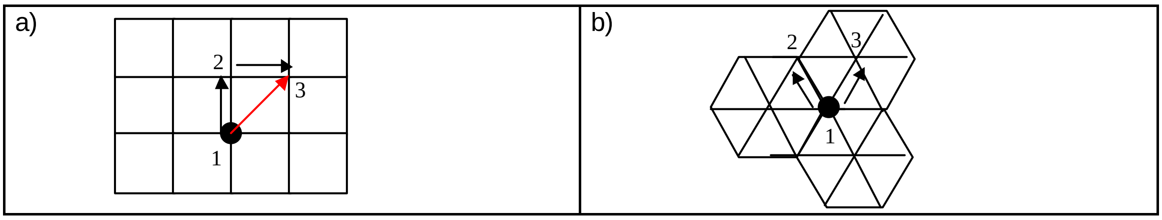


Figure 4 Evolution of the wave front in a square (a) and an hexagonal (b) mesh.

IV NUMERICAL RESULTS

The 2D Elastic Cellular Automaton (ECA) has been used to study the frequency behaviour of the Sierpinski membrane at various orders. The time signal corresponding to the tension at each point mass has been computed and the Fourier transform has been performed. We have limited the calculations to the case of a membrane loaded with negative masses (or holes).

In order to check the validity of the 2D ECA, we have first simulated the Sierpinski membrane order 0 (that is, a uniform membrane). For this case, there is an analytical solution for the eigenfrequencies and the eigenmodes. If the membrane side length is ***a*** and (x, y) are the Cartesian coordinates whose origin is placed at one corner, the transverse deformation $w_{m,n}(x, y)$ corresponding to the ***(n,m)*** eigenmode is given by :

$$w_{m.n}(x, y) = K_{m,n} \cos(\frac{mp}{a} y) \cos(\frac{np}{a} y) .$$

Figure 5 shows the comparison between this analytical solution (constraint given in absolute value) (a) and that obtained by means of the ECA (b).

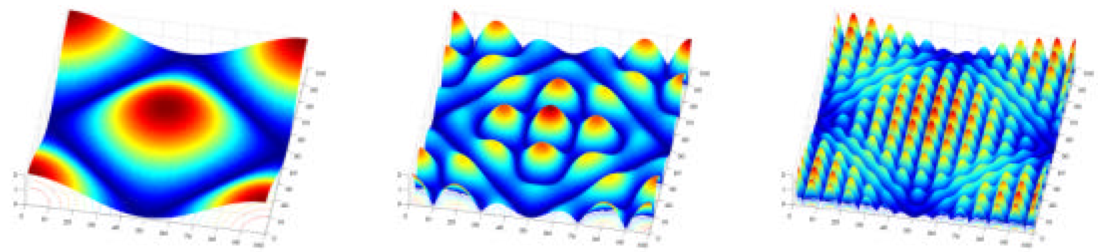


Figure 5(a) : (0,1) (2,3) and (6,7) modal constraints' modes obtained analytically

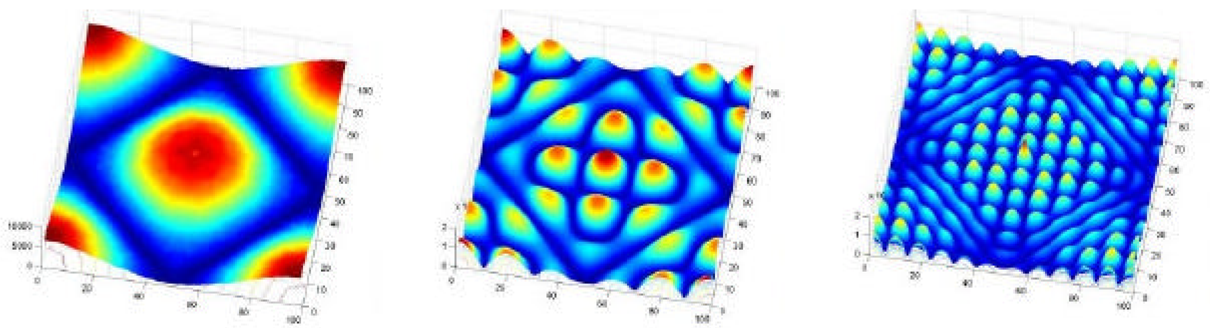


Figure 5(b) : modal constraints obtained with the ECA for three different frequencies

As the results for order 0 are satisfactory from a qualitative point of view, we accept the validity of this approach for higher orders. The following figures (figure 6) show the membrane deformation for order 1 and 2.

a) b)

<u>Figure 6(a)a and 6(b): modal constraints obtained for 551 Hz (order 1) and 547 Hz (order 2)</u>

As we increase frequency from 0 Hz to about 3000 Hz, the 3 structures seems to vibrate the same way and the scatterers seem to have no effect because of their size compared to the wave length. Anyway, the frequencies corresponding to the same mode are a bit lower for order 1 and 2 than for order 0. These frequencies are proportional to the celerity c in a classical membrane, where $c = (T / \rho)^{1/2}$, T is the tension and ρ is the surface weight. By analogy, we can compare scatterers with local weights added to the membrane. The effect of these weight is to increase ρ, so to decrease c and thus to decrease resonance's frequencies.

When the wave length is lower than 5 times the scatterers' size, their effect begins to be important. A part of the n order structure is made with the structure of order n-1 with a scale effect of 1/3 (if we share the structure in 9 equal squares, only the central square is not affected by this transformation). As expected, this scale effect can be observed on the mode's figure : on figure 4, we can see some vibrational modes of structures of order 2 corresponding to a frequency f, and vibrational modes of structures of order 1 corresponding to a frequency f/3. We clearly observe the strong analogy between the figure of n-1 order and one of the 8 peripheral parts of the n order one that we obtain by sharing this figure in 9 equal parts (9 squares). The scale effect is shown here. Anyway, the analogy is not perfect, certainly according to the boundary's conditions that are different in the n-1 order structure, in the corner's parts of the n order structure and in the lateral's parts of the n order structure. We can expect that these differences would decrease if we were increasing the order of the structures, as we shown for the Cantor-like structures that increasing order were decreasing the boundary condition's effects.

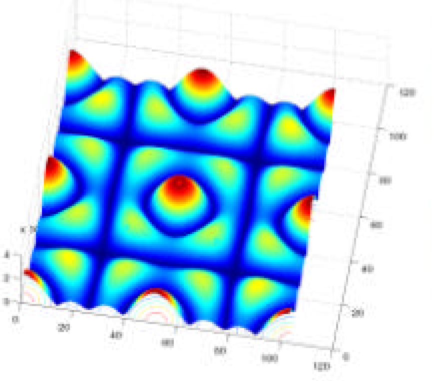
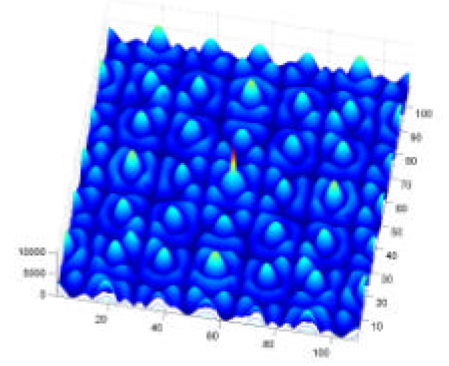
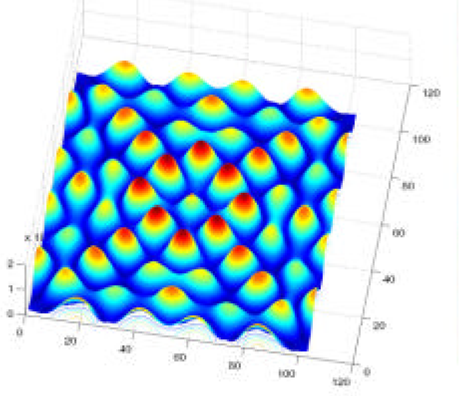
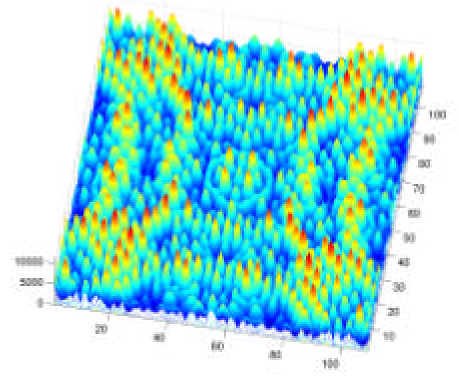
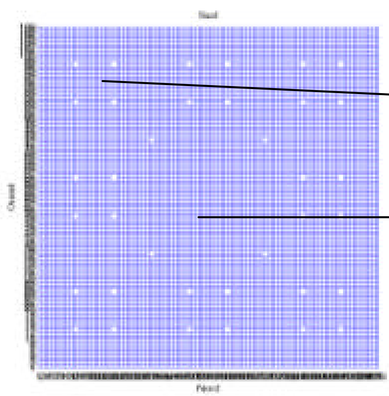
Order 1	Order 2
771 Hz	2313 Hz
	
1203 Hz	3784 Hz
	

Figure 7 : scale effect between two structures' order

Another expected acoustical behaviour of the structure is trapped modes phenomenon : as previously observed with Cantor-like structures, for some frequencies, the multi-reflections of the sound on the scatterers generate resonances at precise places, and destroy the signal everywhere else. On figure 8, such phenomenon is visible and clearly related with the presence of scatterers.



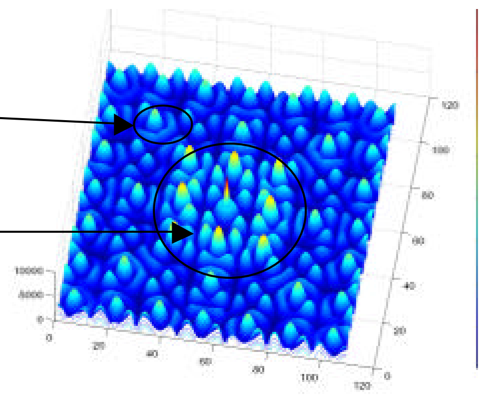


Figure 8 : trapped modes for order 2 at the frequency of 7692 Hz

V CONCLUSIONS

The results obtained for the simulation with a KCA confirm the results obtained with structures generated from the Cantor set (Petri et al [1][2], Gibiat et al [3]) ; scale effects and trapped modes are observed. The use of a cellular automaton instead of a finite difference code gave us to predict some results that have to be confirmed in an experimental way.

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