POWER INJECTED INTO A RECTANGULAR PLATE EXCITED BY A TURBULENT BOUNDARY LAYER

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ABSTRACT

The excitation of structures by turbulent boundary layers has been intensively studied and different models have been established to describe the boundary pressure cross spectrum. When dealing with energy methods like Statistical Energy Analysis (SEA), one is more interested by boundary layer power injected than boundary pressure cross spectrum. This paper deals with derivation of a model for injected power in a plate excited by turbulent boundary layer . For sake of simplicity, it's based on Corcos' model for boundary pressure, but more sophisticated model can be used. The established model of power injection leads to simple analytical expression function of plate and boundary layer characteristics. A comparison of the model with exact calculation is presented.

1. INTRODUCTION

The excitation of a structure by a turbulent boundary layer has been intensively studied. Several models have been established to describe the boundary pressure cross spectrum exciting a rectangular plate. By using experimental results, Willmarth and Wooldridge [1] have described the boundary pressure of a plate excited by a turbulent flow and Corcos [2] has established a model of the boundary pressure cross spectrum. Some models, like the one of Chase [3] or Smol'yakov [4], have been compared in terms of radiated power by Graham [5].

When dealing whith energy methods like SEA, one is more interested in injected power in sub-systems. This paper proposes to establish such a kind of model for injected power. The simplicity of this model shows some interesting properties of the input power.

2. ENERGY TRANSMITTED TO THE PLATE BY THE TURBULENT FLOW

Figure 1 presents the system under study. The plate is of length a, width b ans thickness h. The material has a Young's modulus E, a density ρ and a Poisson's ratio ν. The plate is excited by a turbulent boundary layer, flowing in x direction.

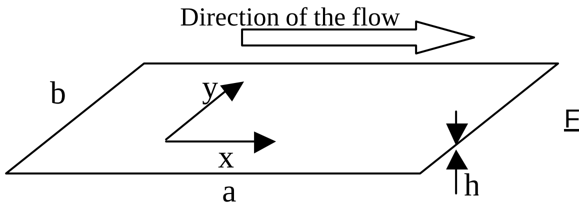

Fig. 1 : Sketch of the system under study

If radiation is not taken into account, the input power is equal to the power dissipated in the plate:

$$P_{inj} = P_{diss} = \eta \omega E(\omega) \tag{1}$$

Where η is the damping loss factor and E(ω) the energy of the plate at angular frequency ω. The energy is related to mean-square velocity of the plate by the following relation:

$$E(\omega) = \rho h \left\langle V^2 \right\rangle \tag{2}$$

If one introduces the displacement spectral density $S_{ww}(\omega)$, the equation (2) becomes :

$$E(\omega) = ab\rho h\omega^2 \overline{S}_{ww}(\omega) \tag{3}$$

To estimate input power, one has to know the displacement spectral density of the plate. Of course, when dealing with experiments, radiation damping exists, it is however often negligible compared to plate loss factor. In the case of calculation, the radiation can be ignored and relation (1) is exact, allowing us to calculate the injected power into the plate by the turbulent boundary layer, from the plate energy. In the following, radiation is not taken into account.

3. CALCULATION OF THE DISPLACEMENT SPECTRAL DENSITY $S_{WW}(\omega)$

The definition of the displacement spectral density is given by the following equation :

$$S_{ww}(x,y,\omega) = W(x,y,\omega)W^*(x,y,\omega) \tag{4}$$

Where W* represents the complex conjugate of W.
One can express the displacement as a modal summation :

$$W(x,y,\omega) = \sum_m \sum_n W_{mn}(x,y)a_{mn}(\omega) \tag{5}$$

Where $W_{mn}(x,y)$ is the mode shape and $a_{mn}(\omega)$ the modal amplitude given by :

$$a_{mn}(\omega) = \frac{F_{mn}(\omega)}{M_{mn}(\omega_{mn}^2 - \omega^2 + i\eta\omega\omega_{mn})} = F_{mn}(\omega)H_{mn}(\omega) \tag{6}$$

Where M_{mn} is the modal mass, $F_{mn}(\omega)$ the generalised modal force and ω_{mn} the natural angular frequency of mode (m,n).

$$M_{mn}\delta_{mnpq} = \int_S \rho W_{mn}(x,y)W_{pq}(x,y)dS \qquad F_{mn}(\omega) = \int_S p(x,y,\omega)W_{mn}(x,y)dS \tag{7}$$

Then, using equations (4), (5) and (6), one can express the displacement spectral density summing modal contributions:

$$S_{ww}(x,y,\omega) = \sum_p \sum_q \sum_m \sum_n H_{mn}(\omega)H^*_{pq}(\omega)W_{mn}(x,y)W^*_{pq}(x,y)I_{mnpq}(\omega) \tag{8}$$

Where I_{mnpq} represents the boundary pressure cross-spectral density of the turbulent boundary layer. Using Corcos' model, one can express I_{mnpq} by the following equation :

$$I_{mnpq}(\omega) = F_{mn}(\omega)F^*_{pq}(\omega) = \int_S \int_S S_{pp}(\xi_1,\xi_2,\omega)W_{mn}(\xi_1)W^*_{pq}(\xi_2)d\xi_1 d\xi_2 \tag{9}$$

Where ξ_1 (resp. ξ_2) are the coordinates (x_1,y_1) (resp. (x_2,y_2)) of excitation points and $S_{pp}(\xi_1,\xi_2,\omega)$ is the boundary pressure cross spectrum. One can obtain mean space value of the displacement spectral density by integrating over the plate surface :

$$\overline{S}_{ww}(\omega) = \frac{1}{ab}\int_0^a \int_0^b S_{ww}(x,y,\omega)dxdy \tag{10}$$

Making use of expression (8), and considering modal orthogonality, one can get for a simply supported plate :

$$\bar{S}_{WW}(w)=\frac{1}{4}\sum_m\sum_n\left|H_{mn}(w)\right|^2 I_{mn}(w) \qquad (11)$$

Finally, using the equations (1) (3) and (11), the input power can be expressed as :

$$P_{inj}(w)=h r h w^3\frac{ab}{4}\sum_m\sum_n\left|H_{mn}(w)\right|^2 I_{mn}(w) \qquad (12)$$

4. THE CORCOS' MODEL

After experiments done by WILLMARTH and WOODRIDGE (1962) [1] on the boundary pressure acting on a rigid plate by a turbulent boundary layer, Corcos has proposed the following law :

$$S_{pp}(x,y,w)=S_{pp}(w)A\left(\frac{wx}{U_c}\right)B\left(\frac{wy}{U_c}\right)e^{-i\frac{wx}{U_c}} \qquad (13)$$

with

$$A\left(\frac{wx}{U_c}\right)=\exp\left(-\frac{1}{a_1}\left|\frac{wx}{U_c}\right|\right) \quad \text{and} \quad B\left(\frac{wy}{U_c}\right)=\exp\left(-\frac{1}{a_2}\left|\frac{wy}{U_c}\right|\right)$$

Where $S_{pp}(\omega)$ is the boundary pressure cross spectrum, x the longitudinal position (direction of the flow), y the tranverse position, U_c the convection speed of the flow and α_1 and α_2 are constants.

Different expressions of the convection speed are given in [8], however in this paper, the simplest model will be used :

$$U_c=KU_\infty \qquad (14)$$

Where K is a constant and U_∞ is the flow speed. In the Corcos' model, U_∞ and $S_{pp}(\omega)$ are obtained from experimental tests.

Aerodynamic coincidence phenomenon occurs when plate bending wave speed is equal to the convection speed U_c. It produces maximum excitation of the plate. The coincidence angular frequency is :

$$w_c=U_c\sqrt{\frac{M}{D}} \qquad (15)$$

Where M is mass per unit area and D the bending stiffness of the plate.

5. CALCULATION OF THE INPUT POWER

Equation (12) gives the expression of the injected power into the plate. In the following, one uses the simplified expression (17) proposed by DAVIES (1971) [7] :

$$I_{mn}(w)=\frac{ab}{4}\frac{S_{pp}(w)}{w^2}a_1a_2U_c^2\frac{1}{1+\left(\frac{a_2U_c}{w}k_n\right)^2}\left[\frac{1}{1+a_1^2\left(1-k_m\frac{U_c}{w}\right)^2}+\frac{1}{1+a_1^2\left(1+k_m\frac{U_c}{w}\right)^2}\right] \qquad (17)$$

Where k_n et k_m are mode wave-numbers : $k_n=\frac{np}{a}$ and $k_m=\frac{mp}{b}$

Finally, one can express the input power by taking into account equations (1), (3), (11) and (17) :

$$\frac{P_{inj}}{S_{pp}(w)}=\frac{hwa_1a_2U_c^2}{rh}\sum_m\sum_n\frac{1}{(w_{mn}^2-w^2)^2+h^2w^2w_{mn}^2}\frac{1}{1+\left(\frac{a_2U_c}{w}k_n\right)^2}\left[\frac{1}{1+a_1^2\left(1-k_m\frac{U_c}{w}\right)^2}+\frac{1}{1+a_1^2\left(1+k_m\frac{U_c}{w}\right)^2}\right] \qquad (18)$$

6. FREQUENCY AVERAGED INJECTED POWER

Energy methods use frequency averaging and do not need the exact knowledge of natural frequencies, but just the number of modes in a frequency band. In this paper, one proposes the derivation of the frequency averaged injected power expression, based on the same kind of assumptions made in energy methods.

The injected power given in equation (18) is averaged on a frequency band 2Δ centered on Ω. The notation for this operation will be :

$$\langle \ \rangle = \frac{1}{2\Delta} \int_{\Omega-\Delta}^{\Omega+\Delta} dw$$

Let us suppose the band of interest is at medium or high frequency and narrow enough to have $\Delta \ll \Omega$. Thus, one can approximate equation (19) by the following expression :

$$\left\langle \frac{P_{inj}}{S_{pp}(w)} \right\rangle = \frac{hwa_1 a_2 U_c^2}{rh} \frac{1}{2\Delta} \sum_m \sum_n \frac{1}{1+\left(\frac{a_2 U_c}{w} k_n\right)^2} \left[\frac{1}{1+a_1^2\left(1-k_m \frac{U_c}{w}\right)^2} + \frac{1}{1+a_1^2\left(1+k_m \frac{U_c}{w}\right)^2} \right] \left\langle \frac{1}{(w_{mn}^2 - w^2)^2 + h^2 w^2 w_{mn}^2} \right\rangle \tag{19}$$

This approximation is justified because, in the band of interest $\Delta \ll \Omega$, the terms in the integral don't varry much except the one representing the modal response for natural frequencies located in the excited band. Moreover, this term produces the dominant contribution of the modal summation.

In order to reduce the calculation to the dominant terms, that is to say to the resonant modes in the band of interest, one can limit the sum to the mode indices m and n verifying :

$$\Omega - \Delta \leq \sqrt{\frac{D}{M}} \left(\frac{n^2 p^2}{a^2} + \frac{m^2 p^2}{b^2} \right) \leq \Omega + \Delta \tag{20}$$

And, as $\Delta \ll \Omega$

$$\Omega \approx \sqrt{\frac{D}{M}} \left(\frac{n^2 p^2}{a^2} + \frac{m^2 p^2}{b^2} \right) \tag{21}$$

Thus, the indices of resonant modes satisfy relation (22) :

$$m = \frac{b}{p} \sqrt{\sqrt{\frac{D}{M}} \Omega - \frac{n^2 p^2}{a^2}} \tag{22}$$

In addition, index of resonant modes exists if :

$$n \leq \frac{a}{p} \sqrt[4]{\frac{M}{D}} \sqrt{\Omega} = n_{\max} \tag{23}$$

One can calculate, with equation (22), the modal density of resonant modes having a fixed index n :

$$n_n(\Omega) = \frac{dm}{d\Omega} = \frac{b}{2p} \sqrt{\frac{M}{D}} \frac{1}{\sqrt{\sqrt{\frac{M}{D}} \Omega - \frac{n^2 p^2}{a^2}}} \tag{24}$$

The number of modes of index n, resonant in the frequency band $[\Omega-\Delta, \Omega+\Delta]$ can be simply calculated :

$$N_n(\Omega) \approx n_n(\Omega).2\Delta \tag{25}$$

Let us approximate equation (19) restricting modal summation to resonant modes.

$$\left\langle \frac{1}{(w_{mn}^2 - \Omega^2)^2 + h^2 w_{mn}^2 \Omega^2} \right\rangle \approx \frac{p}{2} \frac{1}{h^2 w_{mn}^4} \approx \frac{p}{2} \frac{1}{h^2 \Omega^4} \tag{26}$$

Equation (26) approximates the integral from Δ-Ω to Δ+Ω by the integral from 0 to infinity. It's accurate when the pass band of the mode is small compared to the excited band Δ ($2\Delta \gg \eta\Omega$).

Making use of equation (26), one can get equation (27) :

$$\left\langle \frac{P_{inj}}{S_{pp}(\Omega)} \right\rangle = \frac{h\Omega a_1 a_2 U_c^2}{rh} \frac{1}{2\Delta} \sum_n \sum_m \frac{1}{1+\left(\frac{a_2 U_c}{\Omega} k_n\right)^2} \left[\frac{1}{1+a_1^2\left(1-\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} + \frac{1}{1+a_1^2\left(1+\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} \right] \frac{p}{2} \frac{1}{h^2\Omega^4} \quad (27)$$

A second approximation can be done using the number of resonant modes of index n given by expression (25) :

$$\left\langle \frac{P_{inj}}{S_{pp}(\Omega)} \right\rangle = \frac{p a_1 a_2 U_c^2}{2 rh\Omega^2} \sum_n^{n_{max}} \frac{1}{1+\left(\frac{a_2 U_c}{\Omega} k_n\right)^2} \left[\frac{1}{1+a_1^2\left(1-\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} + \frac{1}{1+a_1^2\left(1+\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} \right] \sqrt{\frac{M}{D}} \frac{b}{2p} \frac{1}{\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}} \quad (28)$$

One can notice that this expression is independant of the bandwidth 2Δ. In order to obtain a more compact expression, one can approximate the sum by an integral. Let us also introduce the change of variable n to $k_n = \frac{np}{a}$. One obtain the approximate equation (29) :

$$\left\langle \frac{P_{inj}}{S_{pp}(\Omega)} \right\rangle = \frac{a_1 a_2 U_c^2}{4 rh\Omega^2} \sqrt{\frac{M}{D}} \frac{ab}{2p} \int_0^{k_{n\_max}} \frac{1}{\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}} \frac{1}{1+\left(\frac{a_2 U_c}{\Omega} k_n\right)^2} \left[\frac{1}{1+a_1^2\left(1-\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} + \frac{1}{1+a_1^2\left(1+\frac{U_c}{\Omega}\sqrt{\sqrt{\frac{M}{D}}\Omega - k_n^2}\right)^2} \right] dk_n \quad (29)$$

Changing again of variable k_n to $X = k_n \frac{U_c}{\Omega}$, one obtain :

$$\left\langle \frac{P_{inj}}{S_{pp}(\Omega)} \right\rangle = \frac{a_1 a_2 U_c^2}{4 rh\Omega^2} \sqrt{\frac{M}{D}} \frac{ab}{p} \Psi\left(\frac{w_c}{\Omega}\right) \quad (30)$$

Where

$$\Psi\left(\frac{w_c}{\Omega}\right) = \int_0^{\sqrt{\frac{w_c}{\Omega}}} \frac{1}{\sqrt{\frac{w_c}{\Omega} - X^2}} \frac{1}{1+a_2^2 X^2} \left[\frac{1}{1+a_1^2\left(1-\sqrt{\frac{w_c}{\Omega} - X^2}\right)^2} + \frac{1}{1+a_1^2\left(1+\sqrt{\frac{w_c}{\Omega} - X^2}\right)^2} \right] dX$$

Function $\Psi(\omega_c/\Omega)$ depends only on ω_c/Ω and thus can be calculated numerically as a function of this parameter. Different values are plotted figure 3, using interpolation based on spline function.

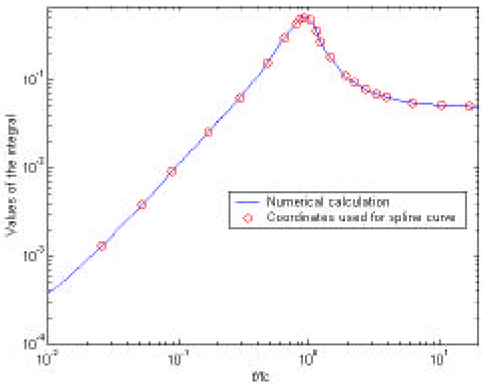

Fig. 3 : Numerical calculation of the integral in function of f/fc

Function $\Psi(\omega_c/\Omega)$ increases before $f/f_c=1$ of 5dB per octave and is constant when f/f_c tends towards infinity. The aerodynamic coincidence phenomenon involves a maximum of the integral.

7. RESULTS AND DISCUSSION

Figure 4 shows the comparison between numerical calculation of the injected power with equation (18) and the calculation of the averaged power with equation (30) :

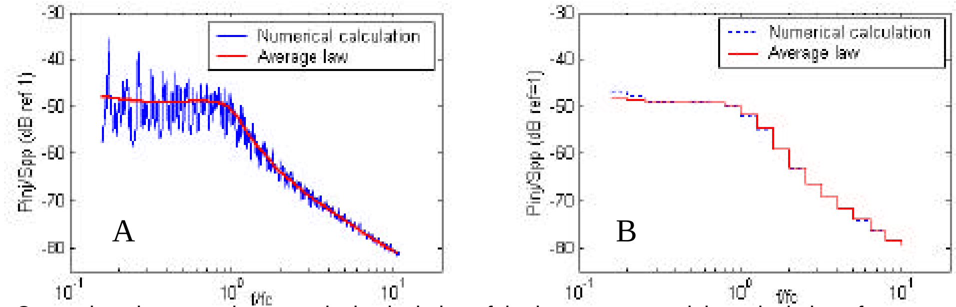

Fig. 4 : Comparison between the numerical calculation of the input power and the calculation of averaged power. Example 1 : a=1m, b=0.7m, h=1mm, U_∞=100ms^{-1}, steel, frequency band 100hz-7000Hz.
B : averaged on 1/3 octave bands.

Results compare very well especially for 1/3 octave band averaging. One can notice that the averaged injected power is almost constant for frequency lower than aerodynamic coincidence frequency f_c and decreases above of 6dB per octave. The aerodynamic coincidence phenomenon does not imply a real increase of injected power. In addition, one can see that power fluctuation with frequency, due to resonances, is related to modal overlap and not to boundary layer excitation.

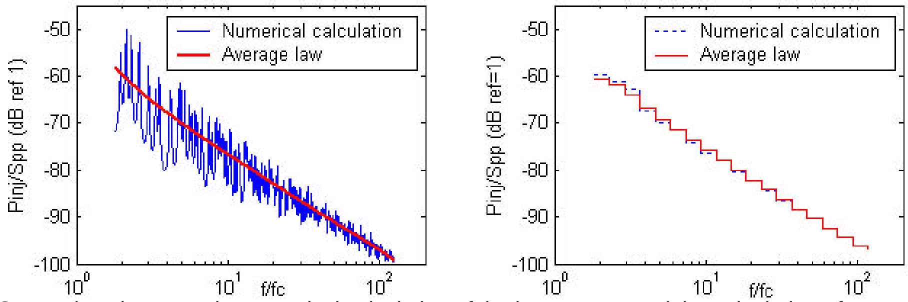

Fig. 5 : Comparison between the numerical calculation of the input power and the calculation of averaged power. Example 2 : a=1.5m, b=0.5m, h=5mm, U_∞=50ms^{-1}, Aluminium, frequency band 100hz-7000Hz
B : averaged on 1/3 octave bands.

This is obvious comparing with example 2. At a given frequency, the plate under study has a lower modal overlap than the one of example 1. Let us consider modal overlap equal to 1, it occurs when f/fc is equal to 0.7 for example 1 and to 62 for example 2. For both cases, fluctuation with frequency of injected power is around 10dB.

CONCLUSIONS

The derived expression for frequency averaged injected power into a plate excited by a turbulent boundary layer compares well with modal calculation. The expression relies on an integral that is only dependant on the ratio ω/ω_c and can be tabulated. Thanks to this formulation, one can easily understand the influence of the characteristics of the plate and the flow on the level of input power. Notably, the frequency averaged injected power is independant of damping.

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