NUMERICAL COMPUTATION OF THE ACOUSTIC PROPERTIES OF POROUS MEDIA OBTAINED BY HOMOGENEISATION TECHNIQUES

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ABSTRACT: In this paper we derive, by using homogeneisation techniques, the effective macroscopic equations of a porous medium composed by a solid skeleton which is completely saturated by a viscous and incompressible fluid. This solid part can be rigid or elastic, which leads to Darcy's or Biot's law, respectively. Apart from providing a rigurous mathematical justification of these models, interest of this methodology is in allowing us to obtain analytical expressions for the averaged parameters characterizing the acoustic behaviour of these porous media. A finite element method is used to calculate these matrix coefficients by solving the corresponding cell problems for a simple pore geometry.

INTRODUCTION

Porous media are frequently met in the nature. In general it is supposed that there are two phases, a solid and a fluid one, where solid can be rigid or deformable.

Modelling of elastic porous materials is a classical issue, undertaken in the extensive works by Biot (see fundamental Biot's papers [1] and [2] or Tolstoy [3] for a recompilation of Biot's works).. Biot´s theory is used to predict the propagation of elasto-acoustic waves through porous media. Materials of this type are actually used in aerospace application, automobile industry or buildings for reducing the noise transmission. For that reasons, the demand of theoretical and numerical tools for a better understanding of the different physical processes and for obtaining accurate predictions has increased considerably during last years.

The mathematical derivation of the model for the macroscopic behaviour of the porous medium is nontrivial since problem is not defined in some fixed domain, but with a sequence of problems in varying geometries. Ene and Sanchez-Palencia [4] seem to be the first to give a derivation for the case of rigid solid using a formal multiscale expansion and beginning from the Stokes system. For the case of a 2D periodic porous medium, Tartar [5] made rigorous this derivation while for the 3D case Allaire [6] did it. Generalisations of these results were given in other papers, like Beliaev and Kozlov [7] for a random statistically homogeneous porous medium. To the contrary, derivation of the model in the case of elastic solid is much less studied. Thus, derivation of the dissipative Biot's law by homogeneisation was studied by a number of authors as Burridge and Keller [8], Sanchez-Palencia [5] or Nguetseng [9], but it has been recently when rigorous derivation from the first principles was made in Gilbert and Mikelić [10] and Clopeau *et al* [11]. The non-dissipative case has been derived in Ferrín and Mikelić [12].

Homogeneisation techniques for the derivation of the macroscopic behaviour of porous media allow us to obtain mathematical expressions for the coefficients in Darcy's and Biot's laws. This can replace the experimental procedures for the determination of these coefficients and aid to porous materials design. We refer to the paper by Biot and Willis [13] to measurement methods for the determination of the elastic coefficients of Biot theory.

In order to describe the acoustic of porous media, among others, Delany and Bazley [14] or, more recently, Allard and Champoux [15] have derived methods to describe sound propagation through rigid porous materials. When elasticity of the skeleton is taken into account and Biot theory is valid, we refer to Allard [16] and references therein.

Concerning numerical simulation of acoustic behaviour of poroelastic materials, Panneton and Atalla [17] have used a 3D finite element formulation in displacements to model a multilayer system, while Atalla *et al* [18] made it with a mixed formulation in displacement and pressure.

In this paper we first show the macroscopic model obtained by homogeneisation, allowing us to get analytical expressions for the coefficients of Darcy's and Biot's laws. Finally, by using a finite element method, we solve the partial differential equations leading to these coefficients which are related to pore geometry and to solid and fluid properties.

DERIVATION OF EFFECTIVE EQUATIONS

The derivation of the macroscopic model for both the case of rigid and the case of elastic solid matrix has been made with the assumption that pores follow a periodic arrangement. Thus, each pore is related to an unit cell, denoted by Y, consisting in a fluid part, Y_f, and a solid part, Y_s. In the two cases studied in this paper, valid in three dimensions, we assume connected solid and fluid parts of the porous media.

Rigid Porous Medium

Beginning from the Stokes system and using a formal multiscale expansion, the first order term leads to the linear relation, known as Darcy's law, linking the seepage velocity, v, with the pressure drop (see [19] for details)

$$v = \frac{K}{m}(f - \nabla p),$$

with K, the permeability tensor, given by

$$K_{ij} = \int_{Y_f} \nabla w^i \nabla w^j \, dy, \quad 1 \le i, j \le 3,$$

where $\{w^i, \boldsymbol{p}^i\}$ is the unique solution of the following problem defined in the fluid part of the periodic cell

$$\begin{cases} -\Delta_y w^i + \nabla_y \boldsymbol{p}^i = e_i & \text{in } Y_f \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f \\ w^i = 0 & \text{on } (\partial Y_f \setminus \partial Y) \\ \int_{Y_f} \boldsymbol{p}^i dy = 0 \end{cases}$$

The model obtained can be considered as a generalisation of that obtained by Allard and Champoux [15], where solid has been supposed isotropic. In [15] the parameter characterising the porous medium is the flow resistivity, σ, while in the model presented in this paper it is the permeability. Relation between them is established as

$$s = \frac{m}{f} K^{-1},$$

where f is the porosity and m is the fluid viscosity.

Elastic Porous Medium

Beginning from the linear elasticity system and Stokes system, one can use the notion of two-scale convergence to obtain the effective equations for this poroelastic medium and to prove the convergence results when the contrast of property number is of order ε^2 (see [11] for details), which means that the normal stresses of the elastic matrix are of the same order as the fluid pressure. The system thus obtained can be compared with Biot's one and coefficients identified. These coefficients can be calculated as averaged values of the solutions of systems of partial differential equations given both in the solid and fluid parts of the cell. More precisely

$$A^H_{klij} = \left(\int_{Y_s} A \left(\frac{e_i \otimes e_j + e_j \otimes e_i}{2} + D_y (w^{ij}) \right) dy \right)_{kl}$$

$$B^H = \int_{Y_s} A D_y (w^0) dy$$

$$C^H_{ij} = \int_{Y_s} \mathrm{div}_{\,y} w^{ij} dy$$

$$\mathrm{A}_{ij}(t) = \int_{Y_f} w_i^j \left(y, \frac{r_f}{m} t \right) dy,$$

which are calculated from the solutions of the following boundary-value problems:

$$\begin{cases} \mathrm{div}_{\,y} \left[A \left(\frac{e_i \otimes e_j + e_j \otimes e_i}{2} + D_y (w^{ij}) \right) \right] = 0 \quad \text{in } Y_s \\ A \left(\frac{e_i \otimes e_j + e_j \otimes e_i}{2} + D_y (w^{ij}) \right) n = 0 \quad \text{on } (\partial Y_s \setminus \partial Y) \\ \int_{Y_s} w^{ij}(y) dy = 0, \qquad w^{ij} \ 1-\text{periodic} \end{cases}$$

$$\begin{cases} -\,\mathrm{div}_{\,y} \left(A D_y (w^0) \right) = 0 \quad \text{in } Y_s \\ A D_y (w^0) n = -n \quad \text{on } (\partial Y_s \setminus \partial Y) \\ \int_{Y_s} w^0 (y) dy = 0, \qquad w^0 \ 1-\text{periodic} \end{cases}$$

$$\begin{cases} \dfrac{\partial w^i}{\partial t} - \Delta_y w^i + \nabla_y \pi^i = 0 & \text{in } Y_f \\ \operatorname{div}_y w^i = 0 & \text{in } Y_f \\ w^i(y,0) = e_i & \\ w^i = 0 \quad \text{on } (\partial Y_f \setminus \partial Y), & \{w^i, \pi^i\}\ 1-\text{periodic} \end{cases}$$

Finally, the Biot system for the diphasic effective behaviour is

$$\bar{\rho}\frac{\partial^2 u}{\partial t^2} - \sum_{i,j} e_i \frac{d}{dt}\int_0^t A_{ij}(t-\tau)\left[\frac{\partial p}{\partial x_j}(x,\tau) + \rho_f \frac{\partial^2 u_j}{\partial \tau^2}(x,\tau)\right]d\tau - \operatorname{div}_x\left[A^H D_x(u)\right] +$$

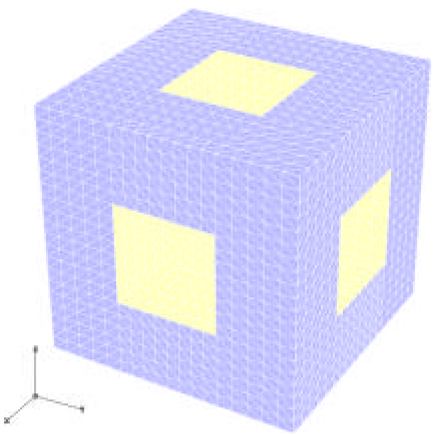
$$+ \operatorname{div}_x\left[\left(\phi I - B^H\right)p\right] = \bar{\rho}F - \sum_{i,j} e_i \frac{d}{dt}\int_0^t A_{ij}(t-\tau)\,\rho_f F_j(x,\tau)\,d\tau$$

$$div_x\left\{\phi\frac{\partial u}{\partial t} + \sum_{i,j} e_i \int_0^t A_{ij}(t-\tau)\left[F_j(x,\tau) - \frac{1}{\rho_f}\frac{\partial p}{\partial x_j}(x,\tau)\right]d\tau - \right.$$

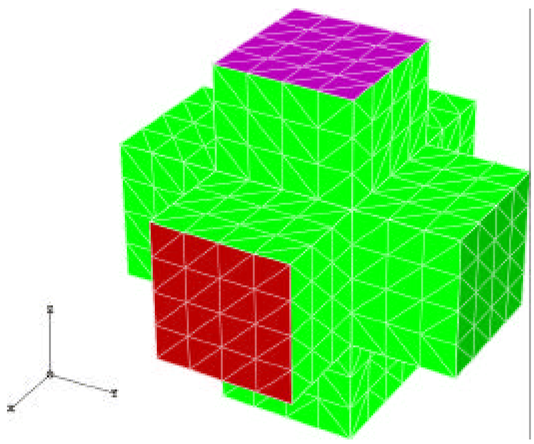
$$\left. - \sum_{i,j} e_i \frac{d}{dt}\int_0^t A_{ij}(t-\tau)\frac{\partial^2 u_j}{\partial \tau^2}(x,\tau)\,d\tau\right\} = C^H : D_x\left(\frac{\partial u}{\partial t}\right) + \frac{\partial p}{\partial t}\int_{Y_s} \operatorname{div}_y w^0\, dy.$$

NUMERICAL RESULTS

The different cell problems written in the previous section have been solved for the following periodic cell, where both fluid and solid parts are shown in next figures



Fluid part of the cell: Y_f



Solid part of the cell: Y_s

For that porous medium we consider that solid is isotropic. The porosity is 0.648 and the physical values considered in this academic test for the fluid and the solid are

Young modulus = 7 10^9	Fluid viscosity = 0.01 $kg\ m^{-1}s^{-1}$
Poisson coefficient = 0.2	Fluid density = 0.2 $kg\ m^{-3}$
Solid density = 400 $kg\ m^{-3}$	

After solving the above equations by using a continuous piecewise linear finite element method in a tetrahedral mesh for the problems in the solid part of the cell and a mixed finite element method for the fluid part, we obtain the following values for coefficients in the macroscopic model, where I is the identity 3x3 matrix:

- Rigid solid part

  Permeability = 0.00285 I

- Elastic solid part

  $\int_{Y_s} \mathrm{div}_y w^0 dy$ = –4.89 10^{-11}

  $C^H = B^H = 0.19\ \mathrm{I}$

  $\mathrm{A}(0) = f\ \mathrm{I}$

  $\mathrm{A}(0.01) = 0.023\ \mathrm{I}$

$$A^H = \begin{pmatrix} 0.17\,10^9\ \mathrm{I} & \mathrm{M}_{12} & \mathrm{M}_{13} \\ \mathrm{M}_{12} & 0.17\,10^9\ \mathrm{I} & \mathrm{M}_{23} \\ \mathrm{M}_{13} & \mathrm{M}_{23} & 0.17\,10^9\ \mathrm{I} \end{pmatrix}$$

where M_{ij} is a symmetric matrix with all zero elements exception made of ij and ji-elements which are 1.7 10^9.

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