Theoretical study of the multiple scattering by a random distribution of dislocations in an elastic medium

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Abstract

The coherent propagation of elastic waves through a (two dimensional) solid filled with randomly placed dislocations is studied in a multiple scattering formalism. Expressions are obtained for the index of refraction and attenuation length (also called elastic mean free path) for both screw and edge dislocations. While the former is easily obtained to first order through elementary arguments, the latter necessitates a careful calculation to second order.

We report here results on the coherent propagation of elastic waves through an elastic medium filled with randomly located dislocations. The behaviour of waves in random media has a long and distinguished history of scholarship and the literature is [1-3]. Current interest stems at least from two sources: the possibility that disorder will induce a change in wave behaviour from transmission to diffusion to localization [4-6], and the enhanced understanding of radiation transfer [7] their study has allowed.

Our motivation for this study, however, comes from the desire to explore possible new non intrusive methods to study the properties of dislocations in materials. There are of course many situations of interest in the study of the mechanical properties of materials where crystal defects play a crucial role and transmission electron microscopy (TEM) appears to be the only technique of choice to characterize such defects in the bulk [8]. Would it be possible to develop new tools? Our results herein suggest acoustic waves could be used as a sensitive probe of dislocation structure. Such a tool would be useful, for instance, to study plastic deformation, current studies thereof underlying how important its fundamental heterogeneity is [9], also to improve current understanding of the brittle-to-ductile transition [10], and of the role played by dislocations in continuous melting [11]. Of course, the interaction of elastic waves with cracks and inclusions in elastic solids has been the object of much attention in the non-destructive-evaluation literature [12].

Topologically, dislocations in an elastic continuum are quite similar to vortex filaments in a fluid, and the coherent propagation of an acoustic wave through a fluid threaded with a random array of vortices has been the subject of a recent investigation [13,14]. The basic mechanism for the scattering of an elastic wave by a line defect is quite simple: An elastic wave will hit each individual dislocation, causing it to oscillate in response. The ensuing oscillatory motion will generate outgoing (from the dislocation position) elastic waves. When many dislocations are present, the resulting wave behaviour can be quite involved because of multiple scattering.
However, under some circumstances, there may exist a coherent wave propagating with an effective wave velocity, its amplitude being attenuated because of the energy being scattered away from the direction of propagation. This is the subject of the present research, in the case of a two dimensional continuum. There are two cases of interest: the anti-plane case, which corresponds to a scalar wave equation for the velocity of the elastic wave in interaction with screw dislocations, and the in-plane case, which corresponds to a vector wave equation in interaction with edge dislocations. The vector nature translates into these waves being a superposition of longitudinal (acoustic) and shear waves.

The anti-plane case - Consider a random distribution of N screw dislocations characterized by their Burgers vectors b^i and their mean positions (i.e., the position they would have in the absence of elastic waves) $\mathbf{X}^i = (X^i_1, X^i_2)$, with i=1, …,N (Figure 1). An elastic wave propagating through such a medium of density ρ and shear modulus μ will be described by a displacement $u(\mathbf{x},t)$ of particles away from equilibrium whose dynamics is best described in terms of velocity $v \equiv du/dt$. Following [15] or [16], it is easy to show that it obeys the equation

$$\left(\rho\frac{\partial^2}{\partial t^2}-\mu\nabla^2\right)v(\mathbf{x},t)=s(\mathbf{x},t)$$

with

$$s(\mathbf{x},t)=\mu\sum_{i=1}^{N}b^i\varepsilon_{ab}V^i_b(t)\frac{\partial}{\partial x_a}\delta\left(\mathbf{x}-\mathbf{X}^i\right)$$

ε_{ab} is defined as $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$. The source term in the right hand side assumes that the dislocation motion is known. In order to have a self consistent formulation we also need the response of a screw to an elastic wave. The response of a dislocation of Burgers vector b located in $\mathbf{X}$ is given in the frequency domain, for low velocities, by [17]

$$V_b(\omega) = -\frac{\mu b}{M\omega^2}\varepsilon_{bc}\frac{\partial}{\partial X_c}v(\mathbf{X},\omega)$$

where M(b) is an effective mass per unit length [17,18].

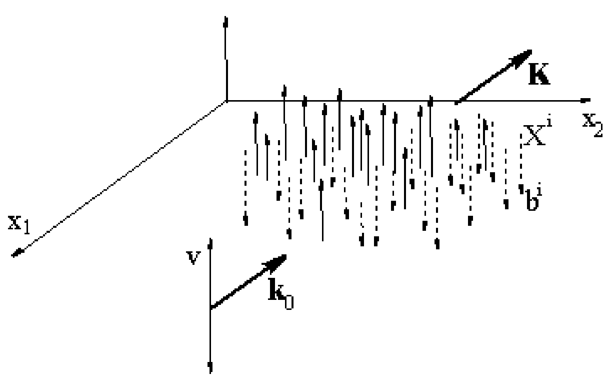

FIGURE 1

A plane wave of frequency ω and wavenumber $\mathbf{k}_0$ traveling through such a medium will propagate coherently with an effective, complex, wavevector $\mathbf{K}$ parallel to $\mathbf{k}_0$ whose real part gives a renormalized speed of propagation, and whose imaginary part gives an elastic mean free path.

In Foldy's approach [1-3], the wave resulting from the scattering by many scatterers is written as the sum of the incident wave v^{inc} and the contribution of the waves scattered by all scatterers receiving themselves the resulting wave

$$v(\mathbf{x}) = v^{inc}(\mathbf{x}) + \sum_{i=1}^{N} F^i v\left(\mathbf{X}^i\right)$$

In this expression, $F^i\, v(\mathbf{X}^i)$ is the wave in $\mathbf{x}$ scattered by the scatterer in $\mathbf{X}^i$; note that $v(\mathbf{X}^i) = \lim_{\mathbf{x} \to \mathbf{X}^i} v(\mathbf{x})$ is singular in this expression. Taking the average of previous equation over all configurations of scatterers leads to

$$\langle v \rangle(\mathbf{x}) = v^{inc}(\mathbf{x}) + n \int d\mathbf{X}\, db\ \rho(b)\ F \langle v \rangle(\mathbf{X})$$

where n is the scatterer density, assumed uniform, and $\rho(b)$ the distribution law for b. In this expression, valid only if chains of scattering paths which go through the same scatterer more than once are neglected, $\langle v \rangle(\mathbf{X})$ is regular. Looking for a solution $\langle v \rangle$ as a plane wave, $F\langle v \rangle$ is then identified to the response v^s of a unique scatterer, located in the origin, to an incident plane wave of amplitude A

$$v^s(\mathbf{x}) = A\ f(\theta)\ \frac{e^{ik_0 x}}{\sqrt{x}}$$

where $f(\theta)$ is the scattering amplitude, with θ the angle between $\mathbf{k}_0$ and $\mathbf{x}$. Equation for s is solved for a unique scatterer in the first Born approximation (in the equation for $V_b(\omega)$, $v = v^{inc}$) and we find

$$f(\theta) = -\frac{\mu b^2}{2M} \frac{e^{i\pi/4}}{\sqrt{2\pi\omega}\,\beta^{3/2}} \cos\theta$$

where β is the shear wave speed, $\beta^2 \equiv \mu/\rho$. We obtain the modified wavenumber

$$K = k_0 \left(1 - \frac{\mu n \langle b^2 \rangle}{2M^* \omega^2} \right)$$

where M^* is a mean effective mass per unit length. Note the dependence on the mean square of the Burgers vector: the sign of the dislocations do not matter. As expected, the group velocity decreases in the presence of the dislocations. However, previous equation does not provide an attenuation length, meaning that a higher order calculation is called for. This is achieved through a Green's function formalism, where the correction to the wave vector is given by a mass operator, the solution to Dyson's equation [5],[13]. In this formalism, the source term is rewritten in the frequency domain by use of a potential operator V

$$\left(\nabla^2 + k_0^2 + V(\mathbf{x},\omega)\right) v(\mathbf{x},\omega) = 0$$

with

$$V(\mathbf{x},\omega)=\frac{\mu}{\omega^2}\sum_{i=1}^{N}\frac{(b^i)^2}{M(b^i)}\frac{\partial}{\partial x_a}\delta(\mathbf{x}-\mathbf{X}^i)\frac{\partial}{\partial x_a}\bigg|_{\mathbf{X}^i}$$

In this expression, $\mathbf{X}^i$ is taken at its mean position. We obtain an integral equation that can be written symbollically as

$$G = G^0 + G^0\, V\, G,$$

where G^0 is the Green's function in the unperturbed medium and G the Green's function modified by the presence of scatterers. Taking the average of this equation over all realizations of V leads to

$$\langle G\rangle = G^0 + G^0 \langle V\, G\rangle,$$

that can be solved when introducing the self- energy operator Σ in the Dyson equation

$$\langle G\rangle = G^0 + G^0\, \Sigma \langle G\rangle,$$

In the Fourier space, the Dyson equation becomes algebraic and can thus be solved for $\langle G\rangle$ when the mass operator $\Sigma(k)$ is known. We deduce Σ using a perturbation expansion [13,14] to second order in the interaction potential V, in the limit of dilute medium

$$\Sigma = \langle V\rangle + \langle V\rangle G^0\langle V\rangle - \langle V G^0 V\rangle,$$

The result for the mass operator is

$$\Sigma(k)=\frac{\mu n\langle b^2\rangle}{M^*\omega^2}k^2\left[-1+\frac{\mu k_0^2}{8rM^*\omega^2}\frac{\langle b^4\rangle}{\langle b^2\rangle}\left(i+\frac{C}{k_0^2\langle b^2\rangle}\right)\right]$$

where r is a constant, of order 1, depending on $\rho(b)$ and C is a numerical constant of order $1/\pi$. Finally, the effective wavenumber is

$$K = k_0\left\{1-\frac{2\mu n\langle b^2\rangle}{M^*\omega^2}\left[1-\frac{\mu k_0^2}{8rM^*\omega^2}\frac{\langle b^4\rangle}{\langle b^2\rangle}\left(i+\frac{C}{k_0^2\langle b^2\rangle}\right)\right]\right\}$$

The in - plane case - The calculations in the in-plane case are quite similar to the anti-plane case but the wave equation becomes vectorial [15]. This case corresponds to the interaction with edge dislocations of Burgers vectors $\mathbf{b}^i$, located in $\mathbf{X}^i$ (Figure 2), with the in-plane wave described by a vector displacement $\mathbf{u}(\mathbf{x},t)$, with velocity $\mathbf{v}$

$$\rho\frac{\partial^2}{\partial t^2}v_a - c_{abcd}\nabla_b\nabla_c v_d = s_a$$

with a = 1,2. $c_{abcd} = \lambda\, \delta_{ab}\, \delta_{cd} + \mu\, (\delta_{ac}\, \delta_{bd} + \delta_{ad}\, \delta_{bc})$ are the elastic constants and

$$s_a(\mathbf{x},t)=\mu\sum_{i=1}^{N}\left(\varepsilon_{ab}b_c^i+\varepsilon_{cb}b_a^i\right)V_b^i\,\frac{\partial}{\partial x_c}\delta(\mathbf{x}-\mathbf{X}^i)$$

Following [17], it is easy to show that the response of a dislocation of Burger's vector **b** (parallel to axis x_1) located in **X** is given, for low velocities, by

$$\mathbf{V}(\omega) = -\frac{\mu \mathbf{b}}{M\omega^2}\left(\frac{\partial v_1}{\partial X_2}+\frac{\partial v_2}{\partial X_1}\right)$$

where v_i indicates $v_i(\mathbf{X},\omega)$ and M is an effective mass per unit length [17,18]. These time elastic waves are a superposition of longitudinal and shear waves, traveling at speeds $\alpha^2 \equiv (\lambda+2\mu)/\rho$ and β^2.

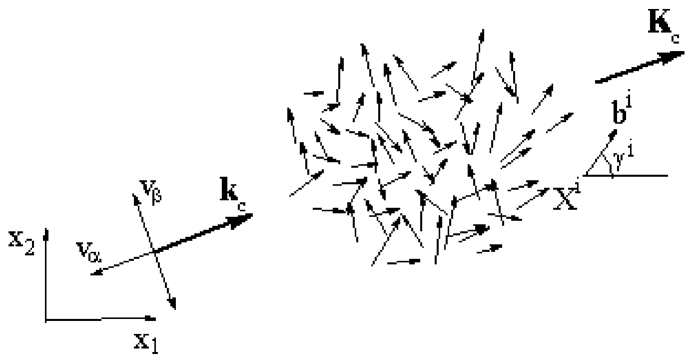

FIGURE 2

The elementary argument to obtain the effective wave velocity of coherent longitudinal and shear waves is best carried out expressing the in-plane displacement in terms of longitudinal φ and shear Ψ potentials. The Foldy's approach may be extended easily to the in-plane case taking into account the mode conversions. The final result is that the plane longitudinal and shear waves with undisturbed wavenumbers k_c ($c = \alpha, \beta$) will propagate coherently with effective wavenumbers

$$K_c = k_c\left(1-\frac{\mu n\langle b^2\rangle A_c}{4M^*\omega^2}\right)$$

with $A_\alpha = \beta^2/\alpha^2$ and $A_\beta = 1$.

Again, it can be noticed that the first order indicates that the group velocity decreases in the presence of dislocations. In order to compute the next order correction an effective Green's function approach is needed, and the corresponding mass operator must be found. In the present case both quantities are rank-two tensors. Calculations are quite similar to the calculations in the scalar case and the end result for the effective wavenumbers K_c ($c = \alpha, \beta$) is

$$K_c = k_c\left\{1-\frac{\mu n\langle b^2\rangle A_c}{4M^*\omega^2}\left[1-\frac{\mu B}{8\rho M^*\omega^2}\frac{\langle b^4\rangle}{\langle b^2\rangle}\left(i+\frac{C}{k_\beta^2\langle b^2\rangle}\right)\right]\right\}$$

where $B = k_\beta^2 + \beta^2 k_\alpha^2/\alpha^2$ and C is a numerical constant of order $1/\pi$.

Conclusion - We have studied the propagation of an elastic wave in a bidimensional medium filled with dislocations (screw or edge) at random. For low densities, we have shown that the solid behaves as an effective homogeneous medium, in which the average (coherent) wave propagates with a renormalized wave speed and an exponentially decaying amplitude. This has been performed using two approaches. The first one following Foldy gives the first order in the development of the disturbed wavenumber. In both anti-plane and in-plane cases, the first order only gives access to the change in the wave speeds. The elastic mean free path is reached thanks to the disturbed Green's function formalism developped at second order. The formalism used in both cases appears to be applicable to structures, such as cracks and grain boundaries, that can be modeled as superpositions of dislocations. Work along these lines is in progress.

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Bibliography

[1] A standard reference is A. Ishimaru, Wave Propagation and Scattering in Random Media, IEEE/OUP, New York (1997).
[2] L. L. Foldy, Phys. Rev. 67, 107 (1945); M. Lax, Rev. Mod. Phys. 23, 287 (1951); M. Lax, Phys. Rev. 85, 621 (1952).
[3] V. Twersky, Proc. Am. Math. Soc. Symp. Stochas. Proc. Math. Phys. Eng., 16, 84 (1964).
[4] P. Sheng, (Ed.), Scattering and Localization of Classical Waves in Random Media (World Science, Singapore, 1990); P. Sheng, Introduction to Wave Scattering, Localization, and Mesoscopic Phenomena} (Academic Press, 1995).
[5] B. Velicky, Sound in granular matter : a case of wave propagation in random media.}, http://parthe.lpthe.jussieu.fr/DEA/velicky.html (1999).
[6] A. Tourin, M. Fink and A. Derode, Waves Random Media, 10 R31 (2000).
[7] L. A. Apresyan and Yu. A. Kravtsov, Radiation Transfer: Statistical and Wave Aspects, (Gordon and Breach, 1996).
[8] D. E. Newbury and D. B. Williams, Acta mater. 48}, 323 (2000).
[9] See N. M. Ghoniem, S.-H. Tong and L. Z Sun, Phys. Rev. B, 61, 913 (2000) and A. Needleman, Acta mater. 48, 105 (2000) for inroads into the literature.
[10] Fracture, Instability Dynamics, Scaling and Ductile/Brittle Behaviour, Material Research Society Symposium Proceedings, MRS, Pittsburg (USA), vol. 409, Edited by R.L. Blumberg Selinger, J.J. Mechosky, A.E. Carlsson and E.R. Fuller (1996).
[11] J. G. Dash, Rev. Mod. Phys., 71, 1737 (1999); F. Lund, Phys. Rev. Let., 69, 3084 (1992); R. Arias and F. Lund, Defect and Diffusion Forum, 150-151, 66 (1997); L. Burakovsky and D. L. Preston, Phys. Rev. E, 63, 067402 (2001).
[12] See Y. C. Angel and A. Bolshakov, JASA, 102, 3290 (1997) and references therein.
[13] F. Lund, Sound Vortex Interaction in infinite Media, to appear in “Sound - Flow interaction”, edited by Y. Aurégan, A. Maurel, V. Pagneux and J.F. Pinton (Lectures Notes in Physics, Springer-Verlag).
[14] M. Baffico, D. Boyer and F. Lund, Phys. Rev. Lett., 80, 2590 (1998). D. Boyer, M. Baffico and F. Lund, Phys. Fluids., 11, 3819 (1999). D. Boyer and F. Lund, Phys. Fluids., 11, 3829 (1999). D. Boyer and F. Lund, Phys. Rev. E, 61, 1491 (2000).
[15] T. Mura, Phil. Mag., 8, 843 (1963).
[16] L. Landau and E. Lifshitz, “Theory of Elasticity” (Pergamon Press, New York, 1959).
[17] F. Lund, J. Mat. Res., 3, 280 (1988).
[18] J. Weertman and J. R. Weertman, ``Moving Dislocations'', in {em Dislocations in Solids, Vol. 3}, edited by R. R. N. Nabarro (North Holland, 1980); J.P. Hirth, Acta Mater. 48(1), 93 (2000).