THE ELECTROMECHANICAL COUPLING FACTOR FOR LONGITUDINAL AND TRANSVERSE PROPAGATION MODES.

PACS: 43.38.Ar

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ABSTRACT

In the IEEE Standard on piezoelectricity the electromechanical coupling factor is defined in static conditions as the ratio between the converted and the total energy involved in a transformation cycle (k_{mat}). In this work we show that this definition can also be extended to dynamic cases: by means of 1–D models we demonstrate that, for loss less piezoelectric elements vibrating in the rod (longitudinal) or width (transverse) mode, the k factor can be defined as the square root of the ratio between the converted electric energy and the kinetic total energy; the obtained results are proportional to the appropriate k_{mat}.

INTRODUCTION

The most important property of a piezoelectric material for practical applications is its ability to generate and to detect stress waves, i.e. to convert electrical energy into mechanical energy and vice versa. As it is well known, the electromechanical coupling factor k fully characterizes this energy conversion, taking the elastic, dielectric and piezoelectric properties into account; but, because these properties are strongly dependent on the electrical and mechanical boundary conditions of the piezoelectric element, many different k factors are defined in literature. More precisely, the k_{ij}, or the so-called material coupling factor (k_{mat}) [1], refers to a fundamentally one-dimensional geometry of the piezoelement and it is related to the converted energy in a static or quasi-static transformation cycle. The effective coupling factor k_{eff} refers to an oscillating specimen of any geometry, but it is not clear in the literature how it is related to the energies involved in a dynamic transformation cycle. Berlincourt et al. gave another definition of the coupling factor in their fundamental work on piezoceramics [2]. This definition can be applied in static and dynamic situations and it can be computed, in principle, for any geometry of the piezoelement, but it does not have a precise physical meaning and its relation with k_{eff} which, among all k factors, is the only one which can easily be measured, is not clear. Finally, in the latest IEEE standard on piezoelectricity [3] it has been suggested to abandon any physical interpretation of this parameter, based on energy interaction, and proposed to define it as a natural combination of elastic, electric and piezoelectric coefficients, as they arise in analytical solutions of electrically driven piezoelements of simple geometry, which can be described by one–dimensional models. In a previous work we demonstrated that, like in static conditions, it is possible to define the k factor as ratio of energies also in dynamic situations [4]. Indeed for a loss less piezoelectric element in free oscillation, mechanically and electrically insulated, the k factor can be defined as the square root of the ratio of the converted electrical energy to the total energy involved in a transformation cycle, i.e. the kinetic energy; the value of the k factor computed in this way coincide with the one obtained using the empirical relation of the effective coupling factor k_{eff} which can be easily measured experimentally. In this work, by means of 1–D distributed models, we demonstrate that these results can be applied to loss less piezoelectric elements vibrating in a longitudinal (for example rod) or a transverse (for example width) mode, and that the obtained results are proportional to the appropriate k_{mat}.

THE DYNAMIC COUPLING FACTOR FOR LONGITUDINAL MODES

Let us consider, a piezoelectric bar of length l along z (or 3 direction), with its end faces electroded and with its cross section A of small diameter, compared with the length (see Fig. 1).

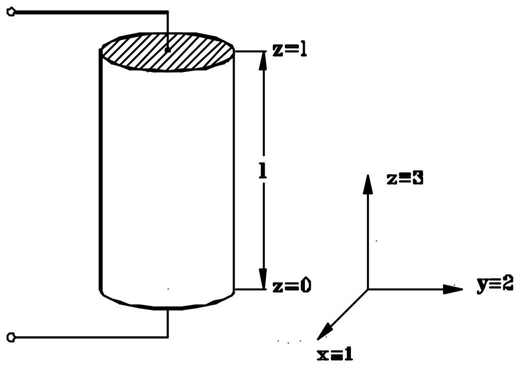

Fig. 1. Geometry of the piezoceramic rod vibrating in a (rod) longitudinal mode.

Suppose that the bar is made of piezoelectric ceramic, which belongs to the 6 mm symmetry class, polarised along the 3 direction. Because of the particular geometry and the electrical and mechanical boundary conditions, the element can be described by a one dimensional stress–strain system and therefore by two scalar constitutive equations [2]:

$$\begin{aligned} T_3 &= \frac{1}{s_{33}^D} S_3 - \frac{g_{33}}{s_{33}^D} D_3 \\ E_3 &= -\frac{g_{33}}{s_{33}^D} S_3 + \beta_{33}^S D_3 , \end{aligned} \qquad (1)$$

where:

$$g_{33} = \frac{d_{33}}{\varepsilon_{33}^T} \; ; \; \beta_{33}^S = \frac{s_{33}^E}{s_{33}^E \varepsilon_{33}^T - d_{33}^2} . \qquad (2)$$

The wave equation can be written as [2]:

$$\rho \frac{\partial^2 \xi_3}{\partial^2 t^2} = \frac{1}{s_{33}^D} \frac{\partial^2 \xi_3}{\partial^2 z^2} , \qquad (3)$$

where ξ_3 is the particle displacement in the z direction and the propagation velocity is:

$$v_3 = \sqrt{\frac{1}{s_{33}^D \rho}} . \qquad (4)$$

By imposing stress–free conditions on the two terminal faces and supposing a sinusoidal excitation ($D_3 = D_0 \bullet e^{j\omega t}$), the solution of the wave equation (3) is:

$$x_3 = g_{33}\frac{v_3}{w}\left[sin\left(\frac{w\,z}{v_3}\right) - tan\left(\frac{J_3}{2}\right)cos\left(\frac{w\,z}{v_3}\right)\right]D_3\,, \tag{5}$$

where $J_3 = wl/v_3$. The particle velocity in the propagation direction z is the time derivative of the displacement:

$$u_3 = j\,g_{33}\,v_3\left[sin\left(\frac{w\,z}{v_3}\right) - tan\left(\frac{J_3}{2}\right)cos\left(\frac{w\,z}{v_3}\right)\right]D_3\,, \tag{6}$$

while the strain along z is the space derivative of x_3:

$$S_3 = g_{33}\left[cos\left(\frac{w\,z}{v_3}\right) + tan\left(\frac{J_3}{2}\right)sin\left(\frac{w\,z}{v_3}\right)\right]D_3\,. \tag{7}$$

The electric field distribution can be computed by the second constitutive equation (1):

$$E_3 = \left\{b_{33}^S - \frac{g_{33}^2}{s_{33}^D}\left[cos\left(\frac{w\,z}{v_3}\right) + tan\left(\frac{J_3}{2}\right)sin\left(\frac{w\,z}{v_3}\right)\right]\right\}D_3. \tag{8}$$

By integrating the electric field along z between 0 and l (see Fig. 1), we obtain the voltage across the two electroded surfaces of the bar:

$$V = D_3\,l\left[b_{33}^S - \frac{2}{J_3}\frac{g_{33}^2}{s_{33}^D}tan\left(\frac{J_3}{2}\right)\right] \tag{9}$$

The kinetic and the potential (converted) electrical energy densities can be computed by using (6) and (9):

$$w_k = \frac{1}{2}\frac{r}{l}\int_0^l |u|^2 dz = \frac{1}{4}\,r\,D_0^2\,g_{33}^2\frac{v_3^2}{J_3}\frac{J_3 - sin\,J_3}{cos^2(J_3/2)} \tag{10}$$

$$w_e = \frac{1}{2Al}C_0\,|V|^2 = \frac{1}{2}\frac{D_0^2}{b_{33}^S}\left[b_{33}^S - \frac{2}{J_3}\frac{g_{33}^2}{s_{33}^D}tan\left(\frac{J_3}{2}\right)\right]^2\,, \tag{11}$$

where $C_0 = A/\left(b_{33}^S\,l\right)$ is the clamped capacitance of the element.

In the already cited paper [4] we demonstrated that, for a loss less piezoelectric element in free oscillation, mechanically and electrically insulated, the k factor can be defined as the square root of the ratio of w_e and w_k. According to IEEE Standard on piezoelectricity [3], this ratio must be computed when the element is fully insulated from the surrounding. The element is mechanically insulated, because we imposed stress free conditions, in order to solve the wave equation (3) and to compute the displacement (5); from an electrical point of view, the boundary conditions impose an exciting current and therefore the element can be considered electrically insulated when it vibrates at a frequency so that the input current goes to zero, or, equivalently, so that the electrical input impedance goes to infinity. The electrical input impedance is given by:

$$Z_i = \frac{1}{j\,wC_0}\left[1 - \frac{2}{J_3}\frac{g_{33}^2}{s_{33}^D\,b_{33}^S}tan\left(\frac{J_3}{2}\right)\right]. \tag{12}$$

As it can be seen from the previous equation, Z_i goes to infinity for $J_3 \to p$, i.e. for ù approaching the antiresonance frequency $ù_0$ = ð v_3 / l and it is evident from (10) and (11) that at this frequency both the kinetic and electric energies go to infinity, in fact we considered a loss less piezoceramic element. In order to obtain the k_w we compute:

$$k_w^2 = \lim_{w \to w_0} \frac{W_e}{W_k} = \frac{8}{p^2} \frac{g_{33}^2}{s_{33}^D b_{33}^S} = \frac{8}{p^2} k_{33}^2 = \frac{8}{p^2} k_{mat}^2 \; . \tag{13}$$

As expected, the value of the dynamic coupling factor (k_w) is less than the value of the static factor (k_{mat}), because in dynamic conditions not all the elastic energy is electrically coupled and vice versa. The proportionality coefficient between k_w and k_{mat} is the same computed by Berlincourt in [1] and relating k_{eff} and k_{mat}; on the other hand, in the already cited paper [4], we demonstrated that the effective coupling factor has the same value of the dynamic factor.

THE DYNAMIC COUPLING FACTOR FOR TRANSVERSE MODES

In order to show that the previous result is independent on the wave propagation direction, we now compute the dynamic coupling factor for a piezoelectric element vibrating in a transverse, for example in the width, mode. Consider a piezoelectric bar with its length w along the x–direction, with the electroded faces normal to the z–direction (the polarisation direction) and with both cross–sectional dimensions a and b small compared with the length (see Fig. 2).

Fig. 2. Geometry of the piezoceramic element vibrating in a transverse (width) mode.

Because of the geometry, the electrical and mechanical boundary conditions, the element can be described by the two scalar constitutive equations [2]:

$$\begin{aligned} S_1 &= s_{11}^E \, T_1 + d_{31} \, E_3 \\ D_3 &= d_{31} \, T_1 + e_{33}^T \, E_3 \, . \end{aligned} \tag{14}$$

The wave equation can be written as [2]:

$$\frac{\partial^2 x_1}{\partial^2 t^2} = v_1^2 \frac{\partial^2 x_1}{\partial^2 x^2} \, , \tag{15}$$

where x_1 is the particle displacement in the x direction and the propagation velocity is:

$$v_1 = \sqrt{\frac{1}{s_{11}^E \, r}} \; . \tag{16}$$

By imposing stress–free conditions on the two faces orthogonal to the x axis (the element is therefore mechanically insulated) and supposing a sinusoidal excitation ($E_3 = E_0 \bullet e^{jw\,t}$), the solution of the wave equation (15) is:

$$x_1 = d_{31} \frac{v_1}{w} \left[sin\left(\frac{w\,x}{v_1} \right) - tan\left(\frac{J_1}{2} \right) cos\left(\frac{w\,x}{v_1} \right) \right] E_3 \, , \tag{17}$$

where $J_1 = w w / v_1$. The particle velocity in the x propagation direction is the time derivative of the displacement:

$$u_1 = j\, d_{31} v_1 \left[sin\left(\frac{w\,x}{v_1}\right) - tan\left(\frac{J_1}{2}\right) cos\left(\frac{w\,x}{v_1}\right)\right] E_3 , \tag{18}$$

while the strain along x is the space derivative of x_1:

$$S_1 = d_{31} \left[cos\left(\frac{w\,x}{v_1}\right) + tan\left(\frac{J_1}{2}\right) sin\left(\frac{w\,x}{v_1}\right)\right] E_3 . \tag{19}$$

The electric displacement can be computed by the second constitutive equation (14):

$$D_3 = \left\{ e_{33}^T \left(1 - k_{31}^2\right) + e_{33}^T k_{31}^2 \left[cos\left(\frac{w\,x}{v_1}\right) + tan\left(\frac{J_1}{2}\right) sin\left(\frac{w\,x}{v_1}\right)\right]\right\} E_3 , \tag{20}$$

where

$$k_{31} = \frac{\mathrm{d}_{31}}{\sqrt{e_{33}^T\, s_{11}^E}} \tag{21}$$

is the static (material) coupling factor for this vibration mode [2].
By differentiating the electric displacement with respect to the time and integrating along the x–y plane (see Fig. 2), we obtain the current in the specimen:

$$I = j\, w\, a\, w\, e_{33}^T \left[1 - k_{31}^2 + k_{31}^2 \frac{2}{J_1} tan\left(\frac{J_1}{2}\right)\right] E_3 . \tag{22}$$

The kinetic energy density can be computed by using the (18):

$$w_k = \frac{1}{2}\frac{r}{l}\int_0^W \left|u_1\right|^2 dx = \frac{1}{4}\frac{d_{31}^2}{s_{11}^E}\frac{J_1 - sin\, J_1}{J_1\, cos^2\left(J_1 / 2\right)} E_0^2 . \tag{23}$$

For this vibration mode the potential (converted) electrical energy must be computed taking the current I into account, because the voltage V represents the electrical excitation:

$$w_e = \frac{1}{2\,a\,b\,w}\,\frac{\left|I\right|^2}{C_{0w}\, w^2} = \frac{1}{2} e_{33}^T \left[1 - k_{31}^2 + k_{31}^2 \frac{2}{J_1} tan\left(\frac{J_1}{2}\right)\right]^2 E_0^2 , \tag{24}$$

where $C_{0w} = e_{33}^T \left(w\, a\right)/\, b$ is the capacitance of the element in Fig. 2.
To compute the k factor we must ensure that the element is also electrically insulated; in this case the electrical boundary conditions impose an exciting voltage and therefore the element can be considered insulated when the input voltage goes to zero, or, equivalently, when the electrical input admittance goes to infinity. The electrical input admittance is given by:

$$Y_i = j\, w C_{0w} \left[1 - k_{31}^2 + k_{31}^2 \frac{2}{J_1} tan\left(\frac{J_1}{2}\right)\right] . \tag{25}$$

In the previous equation it is evident that Y_i goes to infinity for $J_1 \rightarrow p$, i.e. for ù approaching the resonance frequency $ù_0 = ð\, v_1 / w$, as expected, at this frequency both the kinetic and electric energies go to infinity (see (23) and (24)), because we considered a piezoceramic element without losses. In order to obtain the k_w for this vibration mode we, also in this case, compute:

$$k_w^2 = \lim_{w \rightarrow w_0} \frac{w_e}{w_k} = \frac{8}{p^2} k_{31}^2 = \frac{8}{p^2} k_{mat}^2 . \tag{26}$$

Comparing the expression of the dynamic coupling factor computed for this vibration mode with the expression obtained for the longitudinal mode (see (13)), we can observe that the relation between k_w and k_{mat} is the same in both cases. We can therefore conclude that, for a one dimensional vibration mode (longitudinal or transverse), the dynamic coupling factor is related to the so-called material factor by the proportionality coefficient $8/\pi^2$.

CONCLUSIONS

In this paper we have computed the dynamic coupling factor of two piezoelectric elements vibrating in the rod and the width mode respectively. By means of the appropriate 1–D distributed models, the k factor was computed according the definition given in [4], i.e. as the square root of the ratio between the converted electrical energy and the kinetic energy — the total energy involved in a transformation cycle. In both cases we obtained that the dynamic coupling factor is related to the appropriate material coupling factor by the proportionality coefficient $8/\pi^2$. We can therefore conclude that this definition of the dynamic coupling factor can be applied to any one dimensional vibration mode (longitudinal or transverse) of a loss less piezoelectric element. In the future we think to extend this definition to multi dimensional elements and to take the material losses into account.

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