SIMPLIFIED COMPUTATION OF TIME-FREQUENCY DISTRIBUTIONS FOR DOPPLER ULTRASONIC SIGNAL ANALYSIS

PACS: 43.60.Qv, 43.80.Qf

F. García Nocetti, J. Solano González, E. Rubio Acosta and E. Moreno Hernández
Departamento de Ingeniería de Sistemas y Control Automático
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
P.O.Box. 20-726, Delegación Álvaro Obregón, 01000, México, D.F., México
fabian@uxdea4.iimas.unam.mx
julio@uxdea4.iimas.unam.mx
ernesto@uxdea4.iimas.unam.mx
moreno@uxdea4.iimas.unam.mx

ABSTRACT
Conventional methods for signal analysis utilise the Fourier Transform to estimate the spectral response of a signal. However this current practice suffers from poor frequency resolution when estimating non-stationary signals. This paper presents some alternative methods based on time-frequency distributions such as the Wigner Ville, the Choi Williams, the Bessel and the Born Jordan from a Cohen´s class point of view. For each case, a continuous and discrete distribution is formulated, a criterion for determining the interaction between the spectral components of the signal is given and the simplified discretised expression for the calculation of the distribution is proposed that can produce a reduction of at least half of the computations realised when using the original time-frequency distribution definition. A general parallel architecture for the parallel computation of the distribution is also proposed.

INTRODUCTION

A classic method for spectral estimation is the so-called Fourier Transform. However, its use is limited to stationary signals giving as a result poor frequency resolution when estimating non-stationary ones.

There are other spectral estimation methods (Kay, 1988) such as the Periodogram, the autoregressive method, the mobile average and the minimum variance spectral estimation. The performances of these methods also depend on the use of stationary signals and some of them (such as the autoregressive method) utilise short time-segments in order to consider the signal under study as stationary.

Other types of spectral estimators, called time-frequency distributions, have been developed. Unlike conventional methods these distributions are not limited to the use of stationary signals (Cohen, 1989). Despite of this important advantage, the number of calculations involved in obtaining the spectral estimation increases substantially compared to the traditional methods. Therefore, it is desirable to simplify the formulation of the distributions in such a way that the computations involved can be reduced without any loss in the spectral resolution. On the other hand, there are a great variety of time-frequency distributions. It would be very useful to develop an analysis criterion such that can provide a tool for selecting the optimum time-frequency distribution according to the features of the signal under consideration. This paper deals with these issues

TIME-FREQUENCY DISTRIBUTIONS

This section formulates the so-called Cohen's class for the time-frequency distributions and it defines some concepts related to.

The Cohen's Class

The Cohen´s class in terms of time frequency distributions (Cohen, 1989), can be formulated as follows. Let the time-frequency distribution kernel be defined as $\phi(\theta,\tau)$. This kernel will define the particular characteristics of each time-frequency distribution. Let the auto-correlation domain kernel $\psi(t,\tau)$ be defined as the Fourier transform of $\phi(\theta,\tau)$ (from θ to t, considering τ as constant). Let the generalised time-indexed auto-correlation function be defined as

$$R_x'(t,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(t-\mu,\tau)\,x\left(\mu+\frac{\tau}{2}\right)x^*\left(\mu-\frac{\tau}{2}\right)d\mu \qquad (1)$$

Then, the Cohen class for the time-frequency distributions with kernel $\phi(\theta,\tau)$ can be defined as

$$TFD(t,\omega)=\int_{-\infty}^{\infty}R_x'(t,\tau)\,e^{-j\omega\tau}\,d\tau \qquad (2)$$

Auto-term, Crossing Term and Crossing Term Weighting Factor

In order to establish a comparison criterion between the different time-frequency distributions considered in this paper, it is necessary to develop a method that may determine the degree in which the various components of a signal interact when the time frequency distribution is calculated (Cardoso, *et al.*, 1996). Consider the following signal, which is composed of a finite number of sinusoidal signals with constant amplitude A_n, frequency ω_n and phase θ_n

$$x(t)=\sum_{n=1}^{N}A_n e^{j(\omega_n t+\theta_n)} \qquad (3)$$

The n components of the signal (3) interact between them through (1). The interactions of the components with themselves generate the so-called auto-terms of the distribution, which are always positive and constitute the spectral contents of the signal. On the other hand, interactions between different components generate the so-called crossing terms of the distribution, which can be positive or negative and are added to the spectral contents of the signal. Therefore it is desirable to minimise the crossing terms of the distributions.

Substituting equation (3) in (2) and grouping the auto-terms in the first summation and crossing terms in the second summation, a time-frequency distribution can be expressed as

$$TFD(t,\omega)=2\pi\sum_{n=1}^{N}A_n^2\delta(\omega-\omega_n)+\sum_{n=1}^{N}\sum_{\substack{m=1\\ n\neq m}}^{N}F_{TFD}(\omega)A_nA_m\cos\left((\omega_n-\omega_m)t+\theta_n-\theta_m\right) \qquad (4)$$

where $F_{TFD}(\omega)$ is the crossing terms weighting factor, a quantitative measure for evaluating the different time-frequency distributions. The following sections describe the distributions according to their definition.

The Wigner Ville Distribution

According to its definition (Cardoso, *et al.*, 1996; Fan, and Evans, 1994; Martin, and Flandrin, 1985), the Wigner Ville distribution for the continuous case is given by

$$WVD(t,\omega)=\int_{-\infty}^{\infty}x\left(t+\frac{\tau}{2}\right)x^*\left(t-\frac{\tau}{2}\right)e^{-j\omega\tau}\,d\tau \qquad (5)$$

where t is the time and ω is the angular frequency. For the discrete case, the distribution is given by

$$DWVD(n,k)=2\sum_{\tau=-N+1}^{N-1}W(\tau)W^*(-\tau)e^{-j\frac{2\pi k\tau}{N}}x(n+\tau)x^*(n-\tau) \qquad (6)$$

where *n* represents the discrete time and k the discrete frequency. Both variables are normalised.

The Choi Williams Distribution

According to its definition (Cardoso, *et al.*, 1996; Choi, and Williams, 1989), the Choi Williams distribution for the continuous case is given by

$$CWD(t,w)= \int_{-\infty}^{\infty}\sqrt{\frac{1}{4pt^2/s}}\int_{-\infty}^{\infty}e^{-\frac{(t-m)^2}{4t^2/s}}x\left(m+\frac{t}{2}\right)x^*\left(m-\frac{t}{2}\right)dme^{-jwt}\,dt \qquad (7)$$

where $s>0$ is a scaling factor, t is the time and w is the angular frequency. For the discrete case is given by

$$DCWD(n,k)= 2\sum_{t=-N+1}^{N-1}W(t)W^*(-t)e^{-j\frac{2pkt}{N}}\sum_{m=-M}^{M}\sqrt{\frac{1}{4pt^2/s}}e^{-\frac{m^2}{4t^2/s}}x(m+n+t)x^*(m+n-t) \qquad (8)$$

where *n* represents the discrete time and *k* the discrete frequency. Both variables are normalised.

The Bessel Distribution

According to its definition (Cardoso, *et al.*, 1996; Guo, and Durand, 1994), the Bessel distribution for the continuous case is given by

$$BD(t,w)= \int_{-\infty}^{\infty}\frac{2}{pa|t|}\int_{-\infty}^{\infty}\sqrt{1-\left(\frac{t-m}{at}\right)^2}U_0\left(\frac{t-m}{at}\right)x\left(m+\frac{t}{2}\right)x^*\left(m-\frac{t}{2}\right)dme^{-jwt}\,dt \qquad (9)$$

where $a>0$ is a scaling factor, $U_0(t)$ is a second class Chebyshev polynomial, t is the time and w is the angular frequency. For the discrete case is given by

$$DBD(n,k)= 2\sum_{t=-N+1}^{N-1}W(t)W^*(-t)e^{-j\frac{2pkt}{N}}\sum_{m=-2a|t|}^{2a|t|}\frac{1}{pa|t|}\sqrt{1-\left(\frac{m}{2at}\right)^2}x(m+n+t)x^*(m+n-t) \qquad (10)$$

where *n* represents the discrete time and *k* the discrete frequency. Both variables are normalised.

The Born Jordan Distribution

According to its definition (Cohen, 1989), the Born Jordan distribution for the continuous case is given by:

$$BJD(t,w)= \frac{1}{2a}\int_{-\infty}^{\infty}\frac{1}{t}\int_{t-at}^{t+at}x\left(m+\frac{t}{2}\right)x^*\left(m-\frac{t}{2}\right)dme^{-jwt}\,dt \qquad (11)$$

where $a>0$ is a scaling factor, t is the time and w is the angular frequency. For the discrete case is given by:

$$DBJD(n,k)= 2\sum_{t=-N+1}^{N-1}W(t)W^*(-t)e^{-j\frac{2pkt}{N}}\sum_{m=-2a|t|}^{2a|t|}\frac{1}{4a|t|}x(m+n+t)x^*(m+n-t) \qquad (12)$$

where *n* represents the discrete time and *k* the discrete frequency. Both variables are normalised.

EVALUATION OF THE TIME-FREQUENCY DISTRIBUTIONS BASED ON THE CROSSING TERMS WEIGHTING FACTOR.

As stated previously, an ideal crossing terms weighting factor would be one that eliminates the crossing terms. Other desirable situations outside the ideal would be that the weighting factor concentrates the crossing terms due to two different frequency components of a signal around such frequencies and not around other frequencies or spread them out over a wide range of frequencies.

Substituting the signal defined by (3) in (5), (7), (9) and (11) and arranging the crossing terms in the distributions according to (4), the crossing terms weighting factors of each distribution are obtained and defined by the following expressions. For the Wigner Ville distribution:

$$F_{WVD}(w)= 2pd\left(w-\frac{w_n+w_m}{2}\right) \qquad (13)$$

For the Choi Williams distribution:

$$F_{CWD}(w)= \sqrt{\frac{ps}{(w_n-w_m)^2}}e^{-\frac{s}{4(w_n-w_m)^2}\left(w-\frac{w_n+w_m}{2}\right)^2} \qquad (14)$$

For the Bessel distribution:

$$F_{BD}(w) = \frac{4}{a|w_n - w_m|} U_0 \left(\frac{w - \frac{w_n + w_m}{2}}{a(w_n - w_m)} \right) \sqrt{1 - \left(\frac{w - \frac{w_n + w_m}{2}}{a(w_n - w_m)} \right)^2} \qquad (15)$$

For the Born Jordan distribution:

$$F_{BJD}(w) = \frac{p}{a|w_n - w_m|} P_{2a(w_n - w_m)} \left(w - \frac{w_n + w_m}{2} \right) \qquad (16)$$

where $P_a(t)$ is a rectangular symmetrical pulse of duration a. Figure 1 shows a global view of the crossing terms weighting factors for the Wigner Ville, Choi Williams, Bessel and Born Jordan distributions. The weighting factor is defined in terms of w and contains the interacting frequencies w_n and w_m defined in signal (3). Such frequencies can be added or subtracted depending on the behaviour of each distribution. Consider a normalised addition, that is $w_n + w_m = 1$, then the graphs relate the weighting factor against w_n and w, where: $0 < w_n < 1$. Given the normalised addition, for each w_n value, w_m is given by $1 - w_n$.

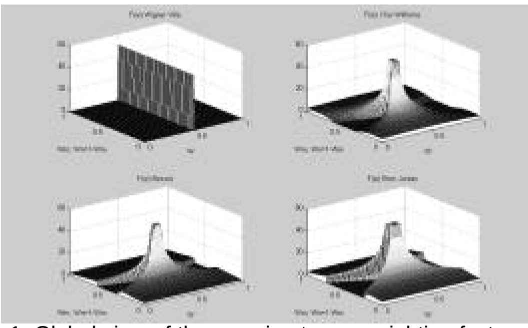


Fig. 1: Global view of the crossing terms weighting factors for the Wigner Ville, Choi Williams $(s = 5)$, Bessel $(a = 2)$ and Born Jordan $(a = 1)$ distributions.

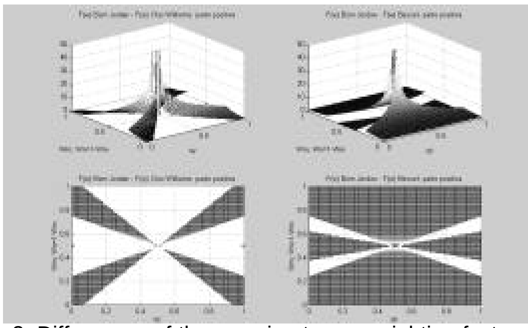


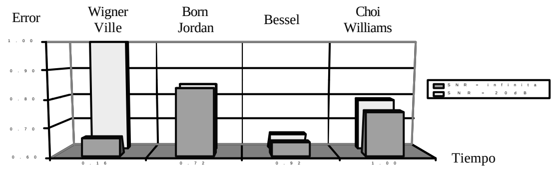
Fig. 2: Differences of the crossing terms weighting factors between Born Jordan-Choi Williams and Born Jordan-Bessel distribution.

ANALYSIS

Figures 2 and 3 depict graphs which show the differences of the crossing terms weighting factors between Born Jordan-Choi Williams, Born Jordan-Bessel, Bessel-Choi Williams and Wigner Ville-any other distribution. The dark zones in the graphs correspond to points where the weighting factor of the first distribution under comparison is greater than the second one. For the purpose of this analysis scaling factors $s = 5$, $a = 2$ and $a = 1$ are considered in the Choi Williams, Bessel and Born Jordan distributions, respectively.

In general, it is observed that the weighting factor for the Bessel distribution is smaller than the Choi William's and the Born Jordan's, and that the Choi Williams's is smaller than the Born Jordan's. These results indicate that the Bessel distribution spreads out the crossing terms better than the Choi Williams's and Born Jordan's distributions and, in consequence, estimates with more precision the spectral contents of a signal in the presence of noise. These results also indicate that the Born Jordan distribution spreads out the crossing terms worse than the Choi Williams's and, in consequence, estimates with less precision the spectral contents of a signal in the presence of noise. All this facts explains the reason why the Bessel distribution is less sensitive to the presence of noise (Cardoso, *et al.,* 1996). But there is a direct relationship between the precision and the amount of calculation involved in the estimation of the spectral contents of the signal.

In the case of the Wigner Ville distribution, the crossing terms are concentrated (due to the two frequency components of the signal) on the average of such frequencies. Therefore, this distribution estimates with better precision the spectral contents of noiseless signals with small bandwidth. The following table shows the instantaneous frequency estimation error versus time elapsed in calculation, both normalised, considering the doppler ultrasonic signal proposed in (Cardoso, 1996).



It is important to point out that all the distributions are affected by a scaling factor which in turn modifies the crossing term weighting factors. As stated previously, the results have been obtained considering $s = 5$, $a = 2$ and $a = 1$ in the Choi Williams, Bessel and Born Jordan distributions, respectively. These optimum scaling factors must be found experimentally and they depend on the characteristics of the signal under study.

REDUCING THE COMPUTATIONAL COMPLEXITY OF THE DISCRETE DEFINITIONS

In order to evaluate the different distributions for spectral estimation, a discrete signal $x(n)$ is

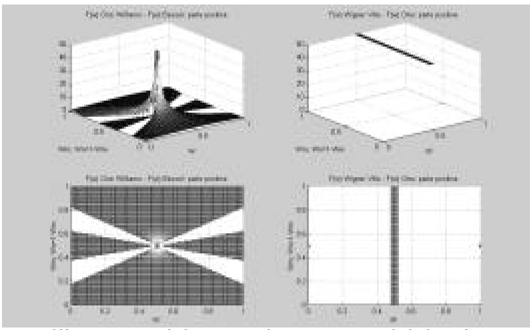


Fig. 3: Differences of the crossing terms weighting factors between Choi Williams-Bessel and Wigner Ville-any other distribution.

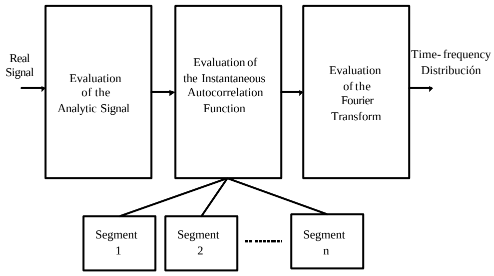


Fig. 4: General Parallel Processing scheme for the evaluation of the time-frequency distributions.

Such a signal contains $2N-1$ elements, where N is a power of 2 and the element range is from $-N+1$ to $N-1$, therefore $x(0)$ is the central element. Based on these elements, this section presents a reduction in computational terms of the number of calculations involved in the evaluation of each of the distributions considered in this paper.

The Wigner Ville Distribution

Considering (6) for estimating the Wigner Ville distribution and evaluating it in n=0 (Boashash, and Black, 1987; Fan, and Evans, 1996), an equivalent simplified expression would be given by

$$DWVD(0,k) = 4\,\mathrm{Re}\left[\sum_{t=0}^{N-1} W(t)W^*(-t)e^{-j\frac{2pkt}{N}} x(t)x^*(-t)\right] - 2|x(0)|^2 \qquad (17)$$

Assuming that $W(t)W^*(-t)$ is a single factor then, for each value of k in (6) evaluated in $n=0$, there are $6N-3$ complex multiplications, $2N-2$ complex additions and 1 scalar multiplication, whereas in (17) there are $3N+1$ complex multiplications, N complex additions and 2 scalar multiplications.

The Choi Williams Distribution

Similarly, considering (8) for estimating the Choi Williams distribution and evaluating it in $n=0$, an equivalent simplified expression would be given by

$$DCWD(0,k) = 4\,\mathrm{Re}\left[\sum_{t=0}^{N-1} W(t)W^*(-t)e^{-j\frac{2pkt}{N}} \sum_{m=-N+1+|t|}^{N-1-|t|} \sqrt{\frac{1}{4pt^2/s}}\, e^{-\frac{m^2}{4t^2/s}} x(m+t)x^*(m-t)\right] - 2|x(0)|^2 \qquad (18)$$

where the summation respect to m for $t=0$ is $x(0)x^*(0)$. Assuming that $M = N-1$ and that $W(t)W^*(-t)$ and $\sqrt{\frac{1}{4pt^2/s}}\, e^{-\frac{m^2}{4t^2/s}}$ are single factors then, for each value of k in (8) evaluated in $n=0$, there are

$8N^2-4N$ complex multiplications, $4N^2-6N+2$ complex additions and 1 scalar multiplication, whereas in (18) there are $2N^2-2N+1$ complex multiplications, N^2-2N complex additions and 2 scalar multiplications.

<u>The Bessel Distribution</u>
Considering (10) for estimating the Bessel distribution and evaluating it in $n=0$, an equivalent simplified expression would be given by

$$DBD(0,k)= 4\,\mathrm{Re}\left[\sum_{\tau=0}^{N-1} W(\tau)W^*(-\tau)e^{-j\frac{2\pi k\tau}{N}} \sum_{m=max\{-2\alpha|\tau|,-N+1+|\tau|\}}^{min\{2\alpha|\tau|,N-1-|\tau|\}} \frac{1}{\pi\alpha|\tau|}\sqrt{1-\left(\frac{m}{2\alpha\tau}\right)^2}\,x(m+\tau)x^*(m-\tau)\right]-2|x(0)|^2 \qquad (19)$$

where the summation respect to m for $\tau=0$ is $x(0)x^*(0)$. Assuming that $W(\tau)W^*(-\tau)$ and $\frac{1}{\pi\alpha|\tau|}\sqrt{1-\left(\frac{m}{2\alpha\tau}\right)^2}$ are single factors then, for each value of k in (10) evaluated in $n=0$, there are $8\alpha N^2-8\alpha N$ complex multiplications, $4\alpha N^2-4\alpha N-2N$ complex additions and 1 scalar multiplication, whereas in (19) there are less than $4\alpha N^2-4\alpha N+1$ complex multiplications, less than $2\alpha N^2-2\alpha N-N$ and 2 scalar multiplications.

<u>The Born Jordan Distribution</u>
Considering (12) for estimating the Born Jordan distribution and evaluating it in $n=0$, an equivalent simplified expression would be given by

$$DBJD(0,k)= 4\,\mathrm{Re}\left[\sum_{\tau=0}^{N-1} W(\tau)W^*(-\tau)e^{-j\frac{2\pi k\tau}{N}} \sum_{m=max\{-2\alpha|\tau|,-N+1+|\tau|\}}^{min\{2\alpha|\tau|,N-1-|\tau|\}} \frac{1}{4\alpha|\tau|}\,x(m+\tau)x^*(m-\tau)\right]-2|x(0)|^2 \qquad (20)$$

where the summation respect to m when $\tau=0$ is $x(0)x^*(0)$. The analysis is similar to Bessel's.

PARALLEL PROCESSING OF THE TIME-FREQUENCY DISTRIBUTIONS.

As stated previously, the use of time-frequency distributions for the spectral estimation of signals opens the possibility of analysing signals that could be non-stationary. However, the computational cost is high. In view of this, this paper has proposed a reduction in the amount of calculations involved for evaluating the original definitions of each distribution, as developed in the previous section. In addition, it is proposed the use of parallel processing techniques to further reduce the time required to perform the evaluations. In particular, a pipeline scheme is used with three stages. The first stage calculates the analytic signal $x_a(t)$ of the real signal. The second stage calculates the generalised time-indexed auto-correlation function $R_x^{'}(t,\omega)$ for $t=0$ of $x_a(t)$. Finally, the third stage calculates the Fourier transform of $R_x^{'}(0,\omega)$, which is the time-frequency distribution $TFD(t,\omega)$ for $t=0$ of the real signal. Figure 4 shows the pipeline structure of the process. For the first and third stages the calculations are relatively simple and a Fast Fourier Transform (FFT) algorithm is used. However, the second stage requires of a more complex process, therefore this stage is further exploited using for this purpose a parallel farm computational model in a star topology. Here, each node calculates a set of operations of the generalised time-indexed auto-correlation function. Although the expressions for the evaluation of each of the time-frequency distributions are different, this second stage can be adapted easily adding or subtracting processors according to the needs. A further analysis of the parallel implementation is a subject of a future work.

CONCLUSIONS

Conventional methods for spectral estimation are limited to the analysis of stationary signals to produce a good estimate. However, these methods offer poor resolution when dealing with non-stationary signals. This paper has presented some alternative methods based on the so-called time-frequency distributions for spectral analysis. Four methods based on the Cohen's class have been analysed, namely the Wigner Ville, the Choi Williams, the Bessel and the Born Jordan distributions. A comparison criterion based on the crossing terms weighting factors has been proposed showing that the Bessel distribution spreads out the crossing terms better than the Choi Williams's and Born

Jordan's distributions and, in consequence, estimates with more precision the spectral contents of a signal in the presence of noise, whereas the Wigner Ville distribution estimates with better precision the spectral contents of noiseless signals with small bandwidth. This analysis has to be conducted taking into account the optimum scaling $s = 5$, $a = 2$ and $a = 1$ for the Choi Williams, Bessel and Born Jordan distributions, respectively. This work also has proposed a simplification in complexity of the expressions utilised for calculating the time-frequency distributions giving as a result a reduction of at least half the operations involved in the original definition. Finally, this paper has proposed a parallel processing scheme for the computation of the time-frequency distribution methods. Here, a pipeline scheme with three stages is utilised, corresponding to the second stage to deal with the more expensive computational process (evaluation of the generalised time-indexed auto-correlation function). A generalised scheme has been described which can adapt easily its topology according to the time-frequency distribution under consideration. Further analysis of the time performance of the system implementation will follow shortly.

ACKNOWLEDGEMENTS

The authors acknowledge the Universidad Nacional Autónoma de México, UNAM (PAPIIT IN106796), CONACYT-REDII(7350-858), CONACYT-México (2146P-A9507) and CONACYT-México (27982-A) for the financial support.

REFERENCES

Boashash, B. and P. Black (1987). An Efficient Real-Time Implementation of the Wigner-Ville Distribution. *IEEE Transactions on Acoustics, Speech, and Signal Processing*. **ASSP-35**. 1611-1618.

Cardoso, J. G. Ruano and P. Fish (1996). Nonstationary Broadening Reduction in Pulsed Doppler Spectrum Measurements Using Time-Frequency Estimators. *IEEE Transactions on Biomedical Engineering*. **43**. 1176-1186.

Choi, H. and W. Williams (1989). Improved Time-Frequency Representation of Multicomponent Signals Using Exponential Kernels. *IEEE Transactions on Acoustics, Speech and Signal Processing*. **37**. 862-871.

Cohen, L. (1989). Time-Frequency Distributions -A Review. *Proceedings of the IEEE*. **77**. 941-981.

Fan, L. and D. Evans (1994). Extracting Instantaneous Mean Frequency Information from Doppler Signals Using the Wigner Distribution Function. *Ultrasound in Med. & Biol*. **20**. 429-443.

Guo, Z., L. Durand and H. Lee (1994). The Time-Frequency Distributions of Nonstationary Signals Based on a Bessel Kernel. *IEEE Transactions on Signal Processing*. **42**. 1700-1707.

Kay, S. (1988). *Modern Spectral Estimation. Theory & Application*. Prentice Hall. New Jersey.

Martin, W. and P. Flandrin (1985). Wigner-Ville Spectral Analysis of Nonstationary Processes. *IEEE Transactions on Acoustics, Speech, and Signal Processing*. **ASSP-33**. 1461-1470.